SELF-EXAMINATIONS

IN

EUCLID.

DESIGNED FOR

SCHOOLS AND UNIVERSITIES.

BY

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"SELF-EXAMINATIONS IN ALGEBRA," &c. &c.

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PREFACE.

THE present Work contains simply the results of experience. During a long series of Geometrical instructions to pupils of various capacities, many ideas have naturally suggested themselves, which the Author, considering valuable, has thought fit to embody in the following Treatise. His primary object has been, to render Euclid easier and more perspicuous; the next, to remedy the defects, and supply the omissions, by which it is vitiated; and the last, to continue the subject of Geometry up to the present advanced state of general Science.

Different Pupils require different helps; one, even of inferior intellect, comprehending a Proposition of more real difficulty, in some instances, with much less trouble than another, of stronger talent, will make out an easier Proposition. It also happens, that the same Pupil at different times requires different assistance; and it may be assumed as a fact generally admitted, that explanations should be various, since what may be obscure even to darkness in one view, may be perfectly clear in another. These considerations have induced the Author to propose a number of Questions relative to the more difficult Propositions, such as he has been in the habit of giving to

his Pupils, and which he apprehends are such as most Tutors find it necessary to give. Euclid, although decidedly the most perfect system of reasoning extant, is yet, in many respects, deficient or erroneous. These defects and errors have been explained, and, in most cases, removed. The text of Simson being taken as the standard, such oversights of that very learned, though partial Editor, as were palpable, have been freely noticed.

ANALYTICAL GEOMETRY having been of late years very properly encouraged in the University, by way of APPENDIX TO THE SELF EXAMINATIONS OF THE TEXT, an introduction to that delightful and comprehensive subject is given. Referring all Points, Lines, Planes, Surfaces, and Solids, or whatever is the subject of Philosophical inquiry, to a certain fixed and definite position, it is commodious to imagine some fixed Point; then, passing through this Point, we conceive three Planes at Right Angles to one another, and perfectly unalterable in position. The positions relative to these Planes of any Magnitude being then given, the position of the Magnitude itself will be given. These Planes, called RECTANGULAR CO-ORDINATE PLANES, are of the most extensive use in Physics, the Positions, and therefore the Motions, or Changes of Positions, of all Bodies, being referred to such Planes in all the best modern Works on Astronomy, Mechanics, and the other branches of the Applied Mathematics. These general views of Science, first introduced by Maclaurin, and subsequently adopted by all the Philosophers of the Continent, and the more enlightened of this Country, have given rise to the

cultivation of ANALYTICAL GEOMETRY according to the method of RECTANGULAR CO-ORDINATES. It is in subserviency to this system of reference of all Points to RECTANGULAR CO-ORDINATE PLANES, that the first APPENDIX has been added.*

This APPENDIX is followed by the famous TREATISE OF PAPPUS ON TANGENCIES, as restored by VIETA. This is extracted from a Work by LAWSON, which is now very scarce, and not likely to be reprinted. The demonstrations are left incomplete, in order to exercise the ingenuity of the Student.

Another APPENDIX, containing DEDUCTIONS FROM EUCLID, has been given, It was the Author's intention at first to have made a Selection of such only as are of use in Physics and Natural Philosophy, and the other branches of Geometry, such as Conics, Spherics, &c. But a slight inspection of the best Works on these branches was sufficient to shew, that much as "Deductions" are cultivated in the University, and despite of the thousands which have teemed from the Press, very few

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* If the reader choose to apply to Mons. Picnot, of 15, Suffolk Street, Pall-Mall East, London, he may be furnished, at a moderate expense, with a set of Co-ordinate Planes, intersecting each other at right angles, and meeting at a common origin. This ingenious artist devised for the Author and a Pupil of his, a mode of actually exhibiting the Co-ordinates of a Point in Space; of Planes, Spheres, Cones, and other Solids in Space, by means of a joint moveable in all directions, something like the Shoulder-joint, which will place the Point, Plane, or Solid, in discussion, in any position that may be required. Students, when furnished with this apparatus, will find the ordinary difficulties attending the representations in solid Geometry, all vanish.

indeed are of any real utility. Those which most frequently occur in Mechanics, Optics, and in "Newton's Principia," have been given at length, retaining the very letters used in the Works referred to. Hence the Student who shall read this Work along with "Euclid's Elements," will have become already familiarised with the principal *Geometrical* difficulties of those profounder Works.

To these Deductions succeed a number of others of inferior importance certainly, but yet not altogether destitute of interest. Some have a value from their properties being very curious; others as affording examples for the methods of investigation. These are proposed for the exercise of the Student's own ingenuity, accompanied, however, with occasional helps and directions.

Many of the Tutors and Lecturers complain against the Works already extant upon Deductions and Geometrical Problems, as calculated to promote a taste in the University for Geometrical quibbling. Impressed with the same sentiments, the Author refrains from pandering to so false an appetite, by supplying his readers with these Mathematical varieties. In reference to similar quirks and conundrums (for they deserve no better name), Simpson has expressed his indignation in no very measured terms, observing, "*there is enough of the useful and practical in Science to exercise the utmost labour and ingenuity of which mankind are capable.*" These remarks are not only applicable to the Geometry which has crept into our system, but also to all other branches in which

this trifling has insinuated itself. In short, instead of spending so many months, if not years, in these unprofitable and therefore insipid speculations, wasting so much valuable time in working the endless combinations of Straight-lines or Circles, or the inexhaustible varieties of Equations or Trigonometrical functions, the Student should dwell upon such questions only as have an intimate connection with the grand and sublime Problems of Nature. With this view all Elementary Works should be composed. All, without exception, should be made preparatory to the great standard and immortal labours of Newton, Lagrange, and Laplace. Were Works thus constructed, Students who are now poring over the diversities of Deductions, or long-winded Algebraical Problems, would be taking up as Class Books the "Mecanique Analytique," or even the "Mecanique Celeste." These observations are corroborated by the opinions and expressions of several of the most distinguished Tutors of the University, of whom one of the most influential has declared, "*that he who shall contribute to the easier comprehension of Laplace, will confer a benefit upon the University at large.*"

In this spirit is composed the following Work. Its intention is to render Euclid easy, and to supply an Introduction to the more comprehensive and powerful Science of ANALYTICAL GEOMETRY.

The Author has availed himself, of course, of the labours of his Predecessors. He has had constantly before him, "Simson's Euclid," "Playfair's, Barrow's, Tacquet's,

De Chales, Elrington's Six Books, Barrow's Geometrical Lectures," "Leslie's Geometry and Geometrical Analysis," &c., &c.; but, although many hints are borrowed from these, especially from the judicious and acute remarks of Elrington, yet the bulk of his labours is drawn from the suggestions of his own experience.

J. M. F. WRIGHT.

Gothic Cottage, Cambridge.

ERRATA.

page	line	for	read
5	*last but one,*	save	have
8	21, 29, *and* 31,	supraposition	superposition
22	24	$\angle H$	$\angle B$
29	20	FG	FS
30	24	CF	CE
34	11	$BCD \perp BC$	$CD \perp AC$
35	8	GB	GD
Also		$\angle^s$	$\triangle^s$
35	25	$<$ the $>$	$< =$ the $>$
36	28	AEH	BEH
38	5	$DF \parallel A$	$DF \parallel AC$
45	6	AD	AD^2
45	14	B	AB
46	*last*	point D	point C
52	20	G and D	G and A
121	1	THEORY	THEORY OF

EXPLANATION

OF

SYMBOLS.

(·)	means	*point*
$\vert$	"	*straight line*
$\Vert$	"	*parallel*
$=$	"	*equal*
$\perp$	"	*perpendicular*
$\odot$	"	*circle*
$\odot^{ce}$	"	*circumference*
▱	"	*parallelogram*
▭	"	*rectangle*
□	"	*square*
$\triangle$	"	*equilateral triangle*
◁	"	*triangle*
$\angle$	"	*angle*
∟	"	*right angle*
$\therefore$	"	*therefore*
$\because$	"	*because* or *since*
$>$	"	*greater than*
$<$	"	*less than*
$\ngtr$	"	*not greater than*
$\nless$	"	*not less than*
$+$	"	*plus* or *added*
$-$	"	*minus* or *less by*
$\sim$	"	*difference of*
$\times$	"	*into* or *multiplied by*
$\div$	"	*divided by*
$\surd$	"	*square root of*

*** Those who object to the introduction of Symbols in Geometry are requested to inspect Barrow's Euclid, Emerson's Geometry, &c., where they will discover many more than are here made use of.

SELF EXAMINATIONS

IN

EUCLID.

DEFINITIONS.

1. *What is a Definition?*

A Definition is that which so describes a thing as to distinguish it from all other things whatsoever.

Thus, if we say that *Man* is a Biped, it is no definition of the animal man, because there are various other Bipeds. If *Man* be described as being that animal which possesses *hands* as well as *feet*, this does not define Man, because monkies and apes have them also; and, moreover, some of us are even born without *hands*. Johnson imagined he had defined Man, when he designated him an 'Animal that cooks its own victuals;' but many animals are there which subject their food to a certain preparation, such as bees, &c. A Definition, it may also be observed, should be couched in no terms incapable of definition, except such as express simple ideas. In what precedes, for instance, 'animal' has not been defined, nor is it easy to discriminate accurately between things animate and inanimate. Again, it may be demanded, what do you mean by 'victuals?' The reply is equally difficult. It requires, indeed, the utmost thought of which the most logical understanding is capable, to avoid these usual errors of definitions.

Definition I.

2. *Does Euclid perfectly define a point?*

By describing it as having no parts or magnitude, he merely asserts that which may be said of other things; for instance, I am now thinking his definition a vague one, but that *thought* has neither parts nor magnitude. Therefore my *thoughts* and a Geometrical *point* are identical.

Playfair defines a point to be 'that which has neither parts nor magnitude, but *position* only.' But what is *position?* It is as difficult to define *position,* as to define a *point.* Since we learn by Astronomy, that every particle of matter is in perpetual motion, who shall say what is Position?

To render this and the second and fifth definition clearer, it is necessary as Simson rightly asserts, to consider a Solid, or that which has length, breadth, and thickness. The boundary of a Solid is a *superficies* having length and breadth, but no thickness; the boundary of a superficies is a *line,* having length, but neither breadth nor thickness; the boundary or extremity of a line is a *point,* which has neither length, breadth, nor thickness.

Definition IV.

3. *What objection can be made to the definition of a straight line?*

In describing it to be 'that which lies *evenly* between its extreme points,' Euclid merely tautologizes, since, 'straight' and 'even' mean the same thing. It is the same as saying, that a *straight line* is an *even line.* What then is an *even line?*

Some, amongst whom is Archimedes, would define a straight line to be the 'shortest of all the lines that can be drawn from one given point to another given point.' But this involves a theorem, viz. 'Of all lines that can be drawn from one point to another, that which is *straight,* is the shortest.'

The very word '*straight*' is, of itself, sufficient to describe the meaning of Euclid; and in fact, a straight line needs no definition. Euclid had just as much reason to define *curve line* as to define *straight line*.

Definition VII.

4. *Does the word* Plane, *suffice to render intelligible the idea of Plane Superficies?*

In common language by *plane*, we mean that which is perfectly level or even. Every superficies is either plane or uneven, and the terms themselves clearly mark the distinction.

Euclid's Definition is a Theorem, viz.

If any two points be taken in a plane superficies, the straight line between them lies wholly in that superficies.

If not, let any part of the straight line lie out of the superficies; therefore the superficies is then uneven; that is, it is not plane, which is contrary to the supposition.

Definition IX.

5. *To comprehend this definition, it is necessary to bear in mind, that the magnitude of an angle, is quite independent of the length of the straight lines which form it.*

The angle *A*, for instance, is greater than the angle *B*.

A pair of compasses will illustrate, very clearly, by opening them wider and wider, the idea of an angle. By these, angles of various sizes may be shewn, although the sides which form them are always of the same length.

Definition XIII.

6. *This definition is needless, 'term,' 'boundary' and 'extremity' being tautology.*

DEFINITION XXX. &c.

7. *Is there any defect in the definition of a square?*

As Euclid makes it a condition that a circle is a plane figure, so must a square, an oblong, a rhombus, a rhomboid, and trapezium, be *plane* figures. Two straight lines which meet together are necessarily in the same plane. All triangles are also of necessity plane figures. But four-sided figures and polygons may not lie wholly in the same plane.

Is any thing superfluous in this definition?

It is sufficient that *one* of its angles be a right angle.

The definition of a square may be amended as follows:

A square is a plane *four-sided figure, having all its sides equal, and* one *angle a right angle.*

POSTULATES.

8. *What is a Postulate?*

A granted feasibility. It is an impossibility to actually draw a straight line, or a circle; for, in the first place, we cannot make a superficies perfectly *plane*, and secondly, all the lines that can be actually exhibited, will have both breadth and thickness. It must, therefore, be granted, that those approximate forms are exact.

AXIOMS.

9. *What is an axiom?*

A self evident truth.

All the axioms, except the twelfth, are evident to common sense. When we come to proposition 29, more will be said on this subject.

PROPOSITIONS.

Consist of Problems and Theorems.

10. *What is a Problem?*

A Problem is a proposition requiring something to be done, as a figure to be described, &c.

11. *What is a Theorem?*

A truth proposed for demonstration.

12. *What are the principal parts in the entire process of establishing a proposition?*

The GENERAL ENUNCIATION;
The PARTICULAR ENUNCIATION;
The CONSTRUCTION;
The DEMONSTRATION.

13. *What is the General Enunciation?*

It is a general verbal description of what the proposition requires to be *done* or *demonstrated,* according as the proposition is a problem or a theorem.

14. *What is the Particular Enunciation?*

It is a particular description of what is required to be done or demonstrated, giving an actual exhibition upon paper of it.

15. *What is the Construction?*

It consists of the description of such lines as are required, besides those of the particular enunciation, by the demonstration.

16. *What is the Demonstration?*

It is the proof, or process of reasoning, whereby the Theorem or Problem is established.

PROP. I.

17. *What is meant by 'given' in Euclid?*

A *point,* is given, when it is supposed to have a certain known fixed position.

A *straight line* is given, as in Prop. I, when its position and length are known.

Besides the word given, in the Data of Euclid, we find the terms 'given in magnitude,' 'given in position,' 'given in species' which are defined in the definitions of the Data.

18. *What is meant by 'finite straight line,' in Prop.* 1*?*

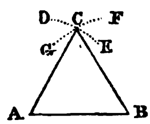
If the straight line were not finite it would not be 'given.' The word is, therefore, superfluous.

19. *Why do the circles necessarily cut one another?*

Because the circumference of one of them passing through the centre of the other, lies partly within and partly without the other. This ought to have been shewn by Euclid. It ought also in this proposition, as well as in the next, to have been stated that the circles and all other lines are in the same plane.

20. *Is it necessary to describe the whole circles?*

In practice it will be sufficient to describe merely those parts which intersect; thus



21. *Is the construction, when the entire circles are described, sufficient for other purposes?*

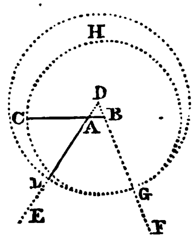
If the lower point of intersection be joined with A and B, another equilateral triangle will be described on the same base AB.

Prop. II.

For 'given point' and 'given straight line,' see 17.

21. *If the given point A were in BC, would the construction be the same?*

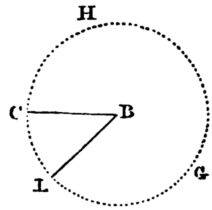
In this case it would be unnecessary to 'join AB.' But in other respects, the Construction and Demonstration will be the same, as is visible by the annexed diagram.



22. *Will Euclid's proof apply to the case when the given point A is in the extremity B?*

When that is the case, there is no need of an equilateral triangle.

Let A coincide with B; then with centre B and distance BC, describe a circle CHG, in CHG take any point L, join BL. BL is the straight line required.



For BL equals BC, and it is drawn from the given point B.

Prop. III.

23. *Is the straight line AB given both in position and magnitude* (see 17).

As the construction makes use of it, that is a necessary condition. At any rate the point A must be given in position, and it must be a condition that AB is greater than C.

24. *What is meant by 'given' as referred to the straight line C?*

Since the straight line C does not enter the construction, it is not necessary that it should be given in position; but in magnitude only.

Prop. IV.

25. *Is any Postulate assumed in the demonstration?*

It is taken for granted, that if two straight lines meet at their extremities, they may be made to lie wholly upon one another to the extent of the less.

Euclid says 'for if the triangle ABC be applied to DEF, so that the point A may be on D, and the straight line AB upon DE, &c.;' which implicates the preceding postulate.

The demonstration may be rendered more perfect, as follows:

For, if the triangle ABC, be applied to DEF, so that the point A may be on D, and the point B on the straight line DE; then AB must lie wholly on DE, for otherwise two straight lines would inclose a space; and AB lying upon DE, B shall coincide with E, because AB equals DE; and AB coinciding, &c.

26. *What sort of demonstration is here employed?*

The application of one triangle to the other is called

Supraposition;

and the absurd, or contradictory result of the argumentation, is demonstrated

Ex absurdo; or, *Reductio ad absurdum;*

in which certain premises being assumed as true, we reason strictly upon them, and yet arrive at a false or absurd conclusion. Hence, we infer that the premises are false, and rectifying them, establish the proposition.

27. *Is it necessary that the supraposition should be actual, or only hypothetical?*

Actual supraposition is impossible from many considerations, an actual triangle being impossible, &c.

28. *Of the three angles and three sides of triangles, how many must be equal, each to each, in order that the triangles may be identical?*

In prop. 4, two sides and the included angle are equal each to each in the triangles.

In prop. 8, the three sides of one triangle are equal to the three sides of the other, each to each.

In prop. 26, two angles and a side adjacent in one triangle, are equal to two angles and a side adjacent, each to each, of the other triangle; as also, two angles and a side are equal to two angles and a side, each to each, the equal sides being opposite to equal angles: in each of which four cases, three of the six angles and sides being equal, the triangles are shewn to be equal and identical in all respects.

In all the six sides and angles, taken three and three together, there are $\frac{6 . 5 . 4}{2 . 3} = 20$ combinations.

Prop. 4 involves three of these combinations.

Prop. 8, only one.

Prop. 26 involves nine.

Consequently, as many as seven of them have not been considered by Euclid.

Of these seven combinations, six of them belong to the case of two triangles, having two sides of the one equal to two sides of the other, each to each, and one angle to one angle, viz. those to which equal sides are opposite.

This case will be fully discussed in art. 96.

The remaining combination, forms a case of itself, which will be considered in art. 97.

Whenever the student is engaged with Prop. 4, 8, or 26, he must bear in mind, that not less than *three* of the six sides and angles of triangles must be equal, each to each, in order that the triangles may be identical.

Prop. V.

29. This proposition is called the 'Pons Asinorum,' but men, of small intellect will easily get over it, if they will proceed 'with measured steps and slow.'

It must first be considered that the point F is to be taken in BD and not in AD, which, as all tutors know, is a common error; then it must be kept in view that the triangles ACF, and ABG are to be proved equal in every respect; then, that $BF = CG$, because the whole AF equal whole AG, and part AB = part AC; then, (making use of what was proved by means of the triangles ACF, ABG) that the triangles BCF, CBG will give

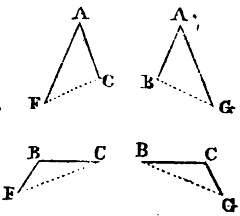
$$\angle\, CBG = \angle\, BCF,$$
$$\text{and } \angle\, CBF = \angle\, BCG;$$

whence the demonstration is easily concluded.

It has proved useful to many students to take the figure to pieces; thus,

First, to prove from the construction and data, that

$$FC = BG;$$
$$\angle\, ACF = \angle\, ABG$$
$$\text{and } \angle\, AFC = \angle\, AGB;$$



and then, by means of the other pair of triangles, to shew that

$$\angle\, FCB = \angle\, GBC$$
$$\text{and } \angle\, CBF = \angle\, BCG, \text{ \&c. \&c.}$$

Prop. V. Cor.

30. *To demonstrate the* Corollary.

Let ABC be an equilateral triangle, it is also equiangular.

$$\because AB = AC;$$
$$\therefore \angle\, C = \angle\, B.$$
$$\text{Again, } \because AB = BC;$$
$$\therefore \angle\, C = \angle\, A;$$
$$\therefore \angle\, A = \angle\, B = \angle\, C,$$

and the triangle is equiangular.

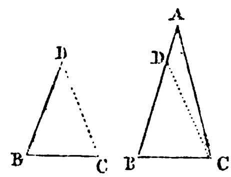
Prop. VI.

The demonstration of this Proposition is *Ex absurdo.*

31. ***What is the difficulty which students find in this proposition?***

It consists in not clearly applying Prop. 4. to the triangles *DBC*, *ACB*. They generally fall into the confused idea of supposing the angle *B*, which is common to the triangles *DBC*, *ACB*, the angle to be used in both triangles; whereas the whole becomes clear when the involved figure is separated, and it is considered that

$$\because DB = AC,$$
$$BC \text{ is common,}$$
$$\text{and included } \angle B = \text{included } \angle ACB;$$
$$\therefore \text{base } DC = \text{base } AB,$$
$$\text{and } \triangle DBC = \triangle ABC \text{ \&c.}$$



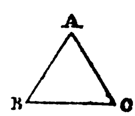
32. ***Why is it said in the demonstration of this proposition, that*** "base *DC* = base *AB*?"

There is no reason for it; excepting, perhaps, that by repeating thus a part of the enunciation of Prop. 4. the proposition itself may be more forcibly brought to remembrance. In strict logic, it is, like some other things that will be remarked upon, quite needless.

Prop. VI. Cor.

33. ***Every equiangular triangle is also equilateral.***

Let *ABC* be an equiangular triangle, it is also equilateral.



$$\text{For } \because \angle A = \angle B;$$
$$\therefore BC = AC;$$
$$\text{Also, } \because \angle C = \angle B$$
$$\therefore AB = AC;$$
$$\therefore AB = AC = BC$$

and the triangle *ABC* is equilateral.

Prop. VII.

This is *Ex Absurdo.*

34. *What is the difficulty of this proposition?*

Case 1. If the student will bear in mind that after having shewn that

$$\angle ACD = \angle ADC$$

he must next affirm that

$$\angle ACD \text{ is} > BCD,$$

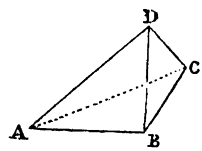
and constantly recollect that

$$AC = AD, \text{ and } BC = BD,$$

the rest of the demonstration will suggest itself to him.

35. *But is it absolutely necessary that*

$$\angle ACD > \angle BCD?$$



If the triangles have the position exhibited in the annexed diagram, the demonstration of the text would not apply, for then

$$\angle ACD \text{ is} < BCD,$$

and the demonstration must be changed to

$$\because AC = AD;$$

$$\therefore \angle ADC = \angle ACD.$$

But if BC fall without the $\angle ABD$, then

$$\angle ADC \text{ is} > BDC;$$

$$\therefore \angle ACD \text{ is} > BDC;$$

much more then is

$$\angle BCD > BDC \text{ \&c.}$$

In short, for the perfection of this proposition, the first case ought to have been demonstrated in two separate cases; first, considering that in which BD falls without the $\angle CBA$, as in Euclid's figure; and secondly, that in which DB falls within the $\angle CBA$.

Students who read Euclid merely as a task will probably exclaim three cases are sufficiently irksome in one proposition, but the lover of strict and perfect reasoning will, in this case, think otherwise.

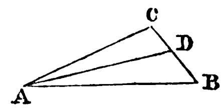
To make the second case easy, it must constantly be remembered that the order is, first to shew that

$$\angle ECD = \angle FDC,$$

and then that

$$\angle ECD \text{ is } > BCD, \text{ \&c.}$$

36. *Why does the third case require no demonstration?*



Because, if the vertex D be in the side BC, of course, unless D coincide with C, and the *two* triangles be *one* only, BD is not equal to BC.

In all the cases (see figure to Art. 35) the sides terminated in one extremity, as A, may be equal, but the two others cannot then also be equal.

37. *Why is it a condition of the enunciation that the triangles are on the same side of the base AB?*

Because, as we shall see by prop. 22, that if we take any triangle on one side of the base, another on the other side may be described so that the two triangles shall have their sides terminated in one extremity of the base equal, and likewise those which are terminated in the other extremity, equal.

Prop. VIII.

38. *What is the defect of this proposition?*

The demonstration ought to have run thus:

'Therefore BC coinciding with EF': if the $\triangle\ ABC$ be on the same side of the base EF with $\triangle\ EDF$, 'BA and AC shall coincide with ED and DF' &c.

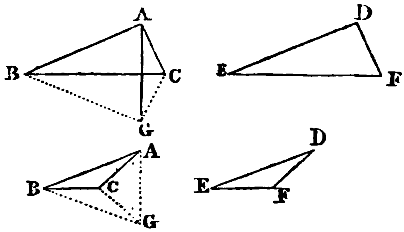
39. *Does this proposition demonstrate that the $\angle A = \angle D$ only?*

Since the triangles wholly coincide, it also follows that

$$\angle B = \angle DEF, \angle C = \angle DFE,$$

$$\text{and } \triangle\ ABC = \triangle\ DFE.$$

40. *To demonstrate Prop. 8 independently of Prop. 7, and without an Ex Absurdo.*



Let ABC, DEF be two triangles having the two sides

$$BA, AC = ED, DF$$

each to each, and also

$$BC = EF$$

then $\angle BAC = \angle EDF$

and the triangles shall be identical in all respects.

For, let the $\triangle DEF$ be applied to the $\triangle ABC$, so that the point E may be on B, and the straight line EF on BC, then F shall coincide with C, because $EF = BC$; and EF coinciding with BC, let D fall on the other side of BC, viz. on G, and join AG.

Then, $\because$

$$BG = ED, \text{ and } AB = ED;$$

$$\therefore BG = AB,$$

$\therefore$ (Prop. 5)

$$\angle BAG = \angle BGA.$$

Again, $\because$

$$GC = DF, \text{ and } AC = DF;$$

$$\therefore GC = AC;$$

$$\therefore \angle CAG = \angle CGA.$$

$$\text{But } \angle BAG = \angle BGA;$$

$\therefore$ the whole or remainder

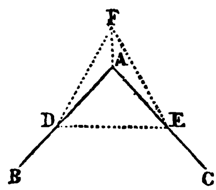
$$\angle BAC = \angle BGC = \angle EDF.$$

If C fall in the straight line AG, the first part of the demonstration is sufficient.

Prop. IX.

41. *Is this proposition defective?*

It ought, for perfect accuracy, to have been a condition, that the $\triangle DFE$ be not on the same side of DE as the $\triangle ADE$.



For, if the construction be as here represented, *AF* does not bisect the ∠ *BAC*, but merely the ∠ *DFE*. If, however, *FA* be produced, it will bisect the ∠ *BAC*; but then Euclid's demonstration will not apply: so that the construction requires thus to be modified, viz.

'Take any point *D* (see figure in the text) in *AB*, and from *AC* cut off *AE* = *AD*, and upon *DE*' and *on the same side of it with B*, 'describe an equilateral triangle,' &c.

42. *Hence an angle may be divided into*

2, 4, 8, 16, 32, &c.

equal angles; viz. by first bisecting it; then bisecting each of the two parts; then bisecting each of the four parts; and so on.

Prop. X.

43. *The word 'finite' is here superfluous, since what is 'given' is necessarily finite.*

44. *Does this proposition include all possible cases?*

If the given point *C* be at an extremity *A*, it will be necessary to use the second postulate by producing the straight line beyond the extremity *C*. The proposition ought, therefore, to have had two cases.

Prop. XI. Cor.

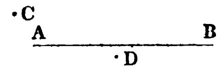
45. *This Corollary is added by Dr. Simson, being, as he conceives, necessary for Prop.* 1 *of Book XI.*

Prop. XII.

46. *Why has the line AB 'unlimited length?'*

Because, if otherwise, the point *C* might have such a position that the circle *ED* would not cut *AB* in two points nor even in one; as in the annexed diagram.

In such a position of C it would be necessary to produce BA, so that AB must be considered as unlimited.



Prop. XIII.

47. Euclid has very properly made two cases of this proposition, for otherwise, the general case which, in the demonstration, involves three angles, could not apply to that case which has only two.

Students in general, find the order of the demonstration of this proposition, difficult to be remembered. That difficulty will be removed if they will only take care to recollect, that the *two* angles CBE, EBD are equal to the *three* angles, and then, that the *two* angles DBA, ABC are equal to the *same three* angles; and therefore that

DBA, ABC are together $=$ CBE, EBD together, &c.

Prop. 15.

48. As in Prop. 13, so in this, if the student take care to recollect that the demonstration chiefly turns upon the first axiom, it will be very easy to him; viz. the angles CEA, AED and AED, DEB are each pair together equal to two right angles, and therefore to one another, &c.

49. *How are the other two vertical angles proved equal?*

The angles AED, AEC and AEC, CEB are each pair together equal to two right angles; and, therefore, (ax. 1) to one another. Take away the common angle AEC, and

remaining angle AED $=$ remaining angle CEB.

Prop. XV. Cor. 1.

50. *Prove the corollary.*

The angles CEA, AED, are equal to the angles BED, BEC, each to each;

$\therefore$ angles $CEA + AED =$ angles $BED + BEC$;

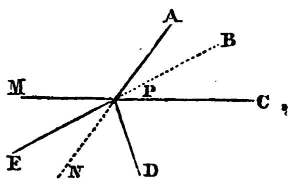
$\therefore$ the four angles are $=$ twice the two angles CEA, AED.

But angles CEA, AED together $=$ two right angles;

$\therefore$ the four angles together $=$ four right angles.

PROP. XV. COR. 2.

51. *Prove the Corollary.*



Let the lines AP, BP, CP, &c. all meet in a point P; the angles which they make at P are together equal to four right angles.

For if any of them, as CP be produced to M, and any other as AP to N; then these angles are together equal to the four angles at the point P made by the straight lines AN, CM; but (Cor. 1) these four angles are together equal to four right angles; therefore the angles made at P by the straight lines AP, BP, CP, &c. are together equal to four right angles.

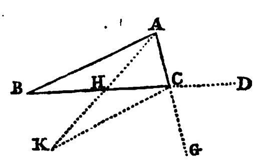
52. *These two Corollaries may both be deduced from Prop.* 13.

PROP. XVI.

53. *Is there any thing superfluous in Euclid's demonstration?*

'Therefore the base AB is equal to the base CF' is needless.

54. *Prove the second case, viz. 'that $\angle$ ACD is also greater than the $\angle$ ABC.'*



Produce AC to G, bisect BC in H, join AH, produce AH making

$$HK = HA,$$

and join KC.

Because in the triangles AHB, KCH

AH, $HB = KH$, HC, each to each,

and the angles included AHB, CHK are equal (Prop. 15)

∴ the angles subtended by equal sides are equal;

$\therefore \angle B = \angle KCH.$

But $\angle GCB$ is greater than the $\angle KCH$;

$\therefore \angle GCB$ is greater than the $\angle B$.

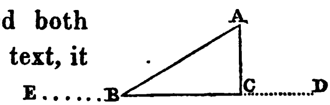
But vert. $\angle ACD$ = vert. $\angle BCG$;

$\therefore \angle ACD$ is greater than the $\angle B$.

Prop. XVII.

55. *Can all three cases of this proposition be demonstrated by one construction?*

If any side as BC be produced both ways to D and E, then as in the text, it may be shewn, that



angles ABC, ACB are together < two right angles.

Again, ∵ $\angle ACD$ is > A to each add the $\angle ACB$;

∴ angles ACD, ACB are together > than the angles A, ACB.

But angles ACD, ACB = two right angles;

∴ angles A, ACB are together < two right angles.

Again, ∵ the $\angle ABE$ is > than the $\angle A$ to each add $\angle ABC$;

∴ the angles ABE, ABC are together > angles A, ABC.

But angles ABE, ABC = two right angles;

∴ angles A, ABC are together < than two right angles.

Prop. XX.

56. *To prove that 'any two sides of a triangle are together greater than the third side,' what is the general construction?*

From the point where they meet, produce either of the two sides and make the part produced equal to the other of the two sides, and join the extremity of the latter side to that of the part produced.

Hence Euclid's construction, would have been better if it had thus been rendered more descriptive, viz.

Produce either of the sides BA, AC, as BA, from A to D, and make AD = the other side AC, and join DC, &c.

The construction being described generally as above, the student encounters no difficulty in demonstrating the other two cases.

Prop. XXII.

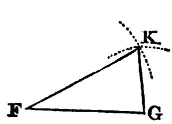
57. *Why must each two of the straight lines A, B, C be greater than the third side?*

Because, if any two of them were not greater than the third, it would be impossible to describe a triangle whose sides shall be A, B, C; for in every triangle any two sides are greater than the third (see Prop. 20).

If, for instance, B and C were not together greater than A or FG, GK were together not greater than FK; that is, if FH were not greater than the semi-diameter, the circle HKL would not cut the circle FKD, and consequently no triangle such as FKG could be constructed.

If FG, GK were together equal to FK, then the points of intersection K, L would merge into one, and the $\triangle$ FKG into a straight line.

58. *What would, in practice, be the simplest mode of solving this problem?*



Take any point F, from which draw FG equal any of the straight lines A, B, C as A. From the centre F at the distance FK equal to either of the other two as B, describe a part of the circumference. Again, from G at the distance C, describe another circumference so as to cut the former in K.

Then, $\because FG = A, FK = B, GK = C$ the triangle is such as was required.

Prop. XXIII.

59. *Required a brief actual construction of an angle equal to the given angle C?*

Take in CD, CE any points D, E and join DE; from AB cut off $AF = DC$, and from A and F, with distances each to each, equal CE, DE describe circles cutting one another in G (see 58) the $\angle A = \angle C$, &c. &c.

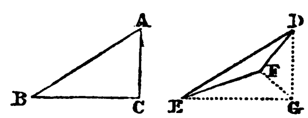
Prop. XXIV.

60. ***Why does Euclid say 'of the two sides DE, DF let DE be the side which is not greater than the other?'***

Because, otherwise, the proposition would consist of two cases.

The first case would be that of the proposition as demonstrated by Euclid, which would apply, when DE is either $>$ or $= DF$.

The second case would be that in which DE is greater than DF, which may be demonstrated as follows.



The construction is the same as Euclid's; which being effected the $\triangle$ DFE is interior to the $\triangle$ DGE, because the $\angle$ EDF is less than the $\angle$ EDG and side DF is less than the side DE; and they are upon the same base DE.

Hence (Prop. 21)

$$DF + FE < DG + GE.$$

But as in Euclid, $DG = AC = DF$, and $GE = BC$;

$$\therefore DF + FE < DF + BC;$$

$$\therefore FE \text{ is } < BC. \quad \text{Q. E. D.}$$

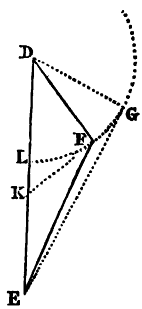
61. This second case, it may be remarked, although more simply demonstrable than by Euclid's method, could not be made to include the case when $DE = DF$.

For then the $\triangle$ DFE would not be interior to the triangle DGE, but they would each be without the other.

62. *In Euclid's construction, ought it not to have been shewn that the points F and D are on different sides of the base EG?*

Some Commentators have remarked upon this omission, but none has supplied it. Amongst others Simson, in his note, observes 'this probably Euclid omitted, as it is easy to perceive that *DG* being equal to *DF*, the point *G* is in the circumference of a circle described from the centre *D* at the distance *DF*, and must be in that part of it which is above the straight line *EF*, *because DG falls above DF, the angle EDG being greater than the angle EDF.*'

Now, that the reasoning here is insufficient, is evident from the annexed construction; in which, although '*G* is in that part of the circumference which is above the staight line *EF*, and *DG* falls above *DF*,' yet the point *F* does not fall below the line *EG*.



63. *For Euclid's demonstration, it is requisite that F fall beyond the straight line EG, or that the ∠ DEF be < than the ∠ DEG; what are then the limits of the construction to which his demonstration will apply?*

Join *FG* and produce it to meet *DE* in *K*. Then the point *F* being in *KG*, if *DE* or *AB* be equal *DK*, it is manifest that *EF* is less than *KG*. Also, if the point *E* fall any where between *K* and *D*, since then the angles *EFD*, *DFG* will together be less than two right angles it is equally clear that *F* will be beyond the line *EG*, and consequently if *E* lie any where between *K* and *D*, Euclid's construction and demonstration hold good; and when *E* falls between *L* and *K* (*L* being in the circumference and *DL* < *DK* because ∠ *DFG* is < right-angle; ∴ *DFK* < right-angle; ∴ *DKF* < right-angle, &c. &c.) — that is, *even in many cases when DE is greater than DF.*

But when E falls below K, as at E in the figure annexed, then the case becomes that demonstrated in Article 60.

From all the comment upon this Proposition, it becomes evident that Euclid had reasons, not perfectly exclusive, for the condition that 'of the two sides DE, DF, let DE be that which is not greater than the other.' For we have seen that innumerable cases exist, viz. those in which E falls between K and L, where his method will apply.

He appears, however, to have adopted the safe side of the limit of his method, by supposing that DE is not to exceed DF or DL; although he ought to have shewn that $\angle\ DEF$ is $>$ than the $\angle\ DEG$.

Prop. XXVI.

64. ***What is the use of that part of the demonstration which states that 'therefore base GC equal base DF?'***

The demonstration is complete without it in Euclid's proof. In the second case it is also needless to say $AH = DF$.

65. ***Why is the $\angle$ BCG less than the $\angle$ BCA?***

Although this is evident by inspection, Euclid has not numbered it among the axioms. It ought, therefore, to have been demonstrated; or which, is better, the conclusion of the demonstration should have been

$\because$ GB, $BC = DE$, EF each to each,
and $\angle\ H = \angle\ E$;
$\therefore$ the angles subtended by equal sides are equal;
$\therefore$ the $\angle\ BCG = \angle\ F$.
But $\angle\ BCA = \angle\ F$;
$\therefore$ $\angle\ BCG = \angle\ BCA$.
But the $\angle\ BCG$ being equal $\angle\ BCA$,
CG shall fall upon CA,
and therefore the point G shall fall on CA.
But it is also in BA;
$\therefore$ it is at their common point A;
$\therefore$ $BG = BA$, or $DE = BA$, &c. &c. Q. E. D.

It may be objected to this that it has neither been assumed in the axioms, nor proved, that two straight lines cannot cut in more points than one; but this follows immediately from axiom 10, because if they could, two straight lines would enclose a space.

66. It also follows from this demonstration that the triangles themselves are equal.

67. From this proposition it immediately follows, that *if from the vertex of an isosceles triangle a perpendicular be let fall upon the base, the perpendicular shall bisect the case and the angle at the vertex;* the converse of which is also evident from Prop. 4.

Prop. XXVII.

68. *What are 'alternate angles?'*

They are the angles which two straight lines make with another at its extremities and on opposite sides of it.

69. *Are the angles AEF, EFD the only two 'alternate' angles?*

The angles BEF, EFC are alternate.

70. *If alternate $\angle\ AEF$ = alternate $\angle\ EFD$, does it necessarily follow that the other two alternate angles are equal?*

Since

$$\angle\ AEF + BEF = \text{two right angles},$$
$$\text{and } \angle\ EFD + EFC = \text{two right angles};$$
$$\therefore \angle\ AEF + BEF = \angle\ EFD + EFC.$$

From these equals take away the equals AEF, EFD, and the remainders BEF, EFC are equal.

Prop. XXVIII.

71. This proposition strictly consists of two. Students usually find something puzzling in it; but the embarrassment

will vanish if they will attend to only one at a time,—first proving that *because the exterior angle is equal to the interior and opposite angle, the straight lines are parallel; and then because the two interior angles are together equal to two right angles that the straight lines are parallel.*

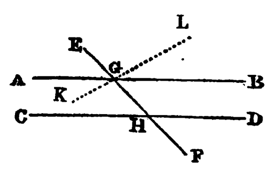
Prop. XXIX.

72. This proposition, which is the converse of Prop. 27, 28, and Axiom 12, upon which it is grounded, has exercised the ingenuity of all Commentators. Some invent new axioms and make axiom 12 a proposition to be proved after Prop. 28, whilst others reject axiom 12 altogether. See Simson's Note.

Legendre proves that the three angles of a triangle are equal to two right angles, and thence deduces this proposition; Simson clears up the obscurity by a long series of definitions and propositions; and Playfair makes it depend upon a new axiom. Of these methods, Playfair's is by far the most simple; the axiom he introduces, viz.

'*Two straight lines which intersect one another, cannot be both parallel to the same straight line,*' being very obvious, and leading to the immediate establishment of the proposition, without any derangement of Euclid's system, or any intermediate Lemmas or introductory propositions.

Playfair's Demonstration.



Let the straight line EF fall upon the parallel straight lines AB, CD; the alternate angles AGH, GHD are equal.

For, if not, let KG be drawn making the $\angle\ KGH = \angle\ GHD$, and produce KG to L; then KL will be parallel to CD; but AB is also parallel to CD; therefore two straight lines are drawn through the same point G parallel to CD, and yet not coinciding with one another, which is impossible. The angles AGH, GHD therefore are not unequal, that is, they are equal, &c.

Prop. XXX.

74. Instead of placing the | EF to which the other two are ||, between them, as in Euclid, it may not be useless if the student vary the position, by making it lie without them.

Prop. XXXII.

75. *What other Corollaries than Euclid's may be deduced from this Proposition?*

3. Any △ can have but one ∟.

4. If a △ have a ∟, then the other two ∠s are together = a ∟.

5. If an isosceles △ have a ∟, each of the = ∠s is half a ∟.

6. Each of the ∠s of an △ is $\frac{1}{3}$ of two ∟s or $\frac{2}{3}$ of one ∟.

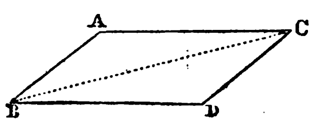
7. If any two △s have two ∠s of the one = two ∠s of the other, *each to each*, the third ∠ of the one is = to the third ∠ of the other.

8. If any two △s have two ∠s of the one *together* = to two ∠s of the other *together*, the third ∠ of the one = the third ∠ of the other.

Prop. XXXIV.

76. In this Proposition the alternate ∠s must be well observed, and also those sides must be considered = which are opposite = ∠s.

77. Prove the converse of this Proposition, viz. first, *that if in any quadrilateral the opposite sides are =, it is a* ▱.



Let $ABCD$ be a quadrilateral having the side AC = side BD, and side AB = side CD; it is a ▱.

Join BC.

∵ in the $\triangle^s$ ABC, ADC,
the sides AB, AC = sides CD, DB each to each,
and their base BC is common.

∴ (Prop. VIII.) $\angle A = \angle D$,
and ∵ $AB, AC = CD, DB$,
and $\angle$ included $A = \angle$ included D.

∴ (Prop. IV.) $\angle^s$ subtended by = sides are =.
∴ $\angle ACB = \angle CBD$,
and $\angle ABC = \angle BCD$.

Again, ∵ the alternate $\angle^s$ ABC, BCD are =,
∴ $AB \parallel CD$.

and ∵ the alternate $\angle^s$ ACD, CBD are =,
∴ $AC \parallel BD$;
and AB was shewn $\parallel CD$.
∴ $ABCD$ is a ▱.

Q. E. D.

Secondly, *If in any quadrilateral the opposite $\angle^s$ are equal, it is a* ▱.

The figure being the same as in the first case, let
$\angle A = \angle D$,
and $\angle ABD = \angle ACD$;
then $ABCD$ is a ▱.

∵ $\angle A = \angle D$,
$\angle ABD = \angle ACD$.
∴ $\angle^s$ A, ABD together = $\angle^s$ D, ACD together.
∴ twice the $\angle^s$ A, ABD = the four $\angle^s$ of the quadrilateral.

But (Cor. 1, Prop. XXXII.),
the $\angle^s$ of a quadrilateral = four $\llcorner^s$.
∴ $\angle^s$ A, ABD together = two $\llcorner^s$.

Hence (XXVIII.), $AC \parallel BD$.

In the same manner it may be shewn that
$AB \parallel CD$.
∴ $ACDB$ is a ▱.

Q. E. D.

From this converse Prop. it appears that the defect of Euclid's Prop. VIII. is not unimportant (see art. 39).

Prop. XXXV.

78. It must be kept in mind here that from the whole trapezium $ABCF$ = △^{s} are to be taken away, leaving = ▱^{s}. This being always kept in view, we are led directly to the △^{s} ABE, DCF, which are proved = by Prop. IV.

Prop. XXXVI.

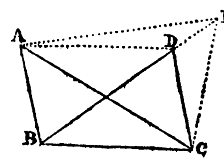
79. The ▱^{s} EG, AC, being both proved = to the constructed ▱ EC, are equal to one another.

Prop. XXXVII.

80. It would be equally correct to complete the ▱^{s} by drawing from C a | $\parallel$ AB, and another from $B \parallel AC$; but the lines intermixing would cause confusion in the figure. This applies to the next Proposition.

Prop. XXXIX.

81. *Is it indifferent whether AE, when drawn $\parallel$ BC, fall on the same or different sides of AD with BC?*



If AE fall above AD, it will be necessary to produce BD to meet AE in E. Then CE must be joined; and following Euclid, it will be proved that △ DCB = △ EBC, the $<$ = the $>$, &c. whereas the contrary is Euclid's conclusion.

It appears, therefore, that there ought to have been two cases.

A similar defect is in the demonstration of Prop. XL.

Prop. XLI.

82. From this Proposition it is easy to see that

If a ▱ and △ are between the same $\parallel^s$, have their

bases upon the same |, and are situated on the same side of it, and the base of the △ be double that of the ▱, the ▱ equals the △.

Also if a ▱ and △ be upon *equal* bases on the same | and on the same side of it, the ▱ is double of the △.

Prop. XLIV.

83. ***How are BE and BG actually determined?***

By first bisecting a side of the △, &c. as in Prop. XLII. which will give a ▱ ≐ the △, and then producing *AB*, and making *BE* = one side of the ▱, and at the (·) *B* making the $\angle$ *EBG* = $\angle$ *D*, and *BG* = the other side of the ▱, which with the former makes that $\angle$ which was made = $\angle$ *D*; finally, by drawing from *E* and *G*, *EF* and *FG* ∥ *BG* and *BE* respectively. It would require proof that *BF* thus described is = to the ▱ described by Prop. XLII. All this trouble, however, Euclid avoids, by supposing the ▱ first made by Prop. XLII., and then transferred and so posited that a side *BE*, containing with *BG* an $\angle$ = $\angle$ *D*, "may be in the same line with *AB*."

This statement of Euclid's, it is evident, is not perfect; because, if *BE* partly coincided with *BA*, and Euclid has not verbally prohibited such a position, the construction would not solve the problem. *BE* must lie in *AB* produced. Moreover, in tacitly supposing the ▱ made equal to the △ *C* to be transferred to *AB*, Euclid assumes a new postulate.

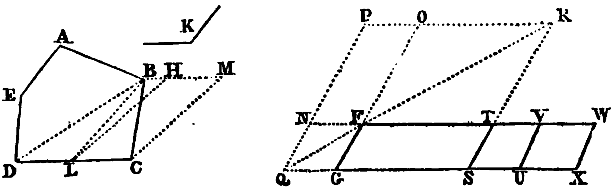
Prop. XLV.

84. This Proposition is not a favourite with the generality of students, because of the tediousness of proving *KH*, *HM* in the same |, and *FG*, *GL* in the same |; it would have been easier, if, after Euclid had shown that *KH* and *HM* are in the same |, he had proceeded thus: *and* ∵ *FG*, *GL*

are each ∥ KM, they are ∥ one another, and being joined at G, are ∴ in the same |, &c.

Prop. XLV. Cor.

85. *Required to apply to a given* | *a* ▱ *which shall have an* ∠ = *a given* ∠ *and shall be* = *a given rectilineal figure?*



Let *ABCDE* be the given rectilineal figure, *K* the given ∠, and *GF* the given |; it is required to apply to *GF* a ▱ equal to the given figure, and having an ∠ = ∠ *K*.

Divide *ABCDE* into △ˢ by joining *AD*, *BD*. Bisect *DC* in *L*, and at the (·) *C* in the | *CD* make the ∠ *LCM* = ∠ *K*, and complete the ▱ *CH*. Produce *GF* to *O* making *FO* = *CM*, at *F* in *FO* make ∠ *OFN* = ∠ *K*, and make *FN* = *CL* and complete the ▱ *FP*. Complete the ▱ *PS* as in Prop. XLIV.

If *NO* and *LM* be joined, it will easily be proved that the △ˢ *LMC*, *NOF* are = and ∴ that the ▱ *PF* = ▱ *CH* = △ *BDC*.

But the complement *FS* = comp. *PF*,
▱ *FG* = △ *DBC*.
and ∠ *GFT* = ∠ *NFO* = ∠ *K*.

Again to *ST* apply the ▱ *TU* so that the ∠ *STV* = ∠ *K*; and to *VU* apply the ▱ *VX*, so that the ∠ *UVW* = ∠ *K*. Then *FX* will be the ▱ required.

It must first be proved that *FT*, *TV*, *VW* are in the same |, and also that *GS*, *SU*, *UX* are in the same |, &c. &c.

Prop. XLVI.

86. Since AC is required to be drawn $\perp$ AB at its extremity, the value of the remarks in art. 44 is by this Proposition made evident.

After AD is drawn $\perp$ and $=$ to AB, it must be remembered that from D and B the other sides are drawn *parallel* to others, &c.

Prop. XLVI. Cor.

87. Let of the ▭ $ABED$ one $\angle$ as A be ⊾, then all its $\angle$ˢ are ⊾ˢ.

$\because$ AE is a ▭, and AD meets the $\|$ˢ DE, AB,

$\therefore$ $\angle$ˢ EDA, DA, B = two ⊾ˢ (XXVIII.)

But $\angle$ DAB = a ⊾,

$\therefore$ $\angle$ EDA = a ⊾.

And the $\angle$ˢ opposite to these being equal to them are also ⊾ˢ. $\therefore$, &c.

Prop. XLVII.

88. *How is the □ CH proved equal to the ▭ CL?*

$\because$ BAC, CAH are each ⊾ˢ,

$\therefore$ BA and AH are in the same | .

Again, $\because$ $\angle$ KCA = $\angle$ ECB, both being ⊾ˢ,

to each add $\angle$ BCA.

$\therefore$ whole $\angle$ KCB = whole $\angle$ FCA.

Also $KC = CA$, and $BC = CE$;

$\therefore$ in the △ˢ ACE, BCK, two sides of the one CK, CB = two sides of the other AC, CE, each to each, and the $\angle$ˢ included by these = sides are =, viz. KCB = ACE.

$\therefore$ △ KCB = △ ACE.

But the □ CH, and the △ KBC, being upon the same base CK, and between the same $\|$ˢ CK, BH, the □ is double of

the △. Also the ▱ CL is double of the △ ACE, because they are upon the same base CE and between the same ‖ˢ CE, AL. But the doubles of equals are equal.

$$\therefore \text{ the } \square\ CH = \text{▱}\ CL, \text{ \&c. \&c.}$$

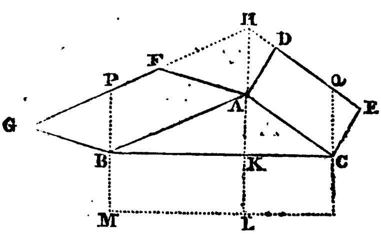
Q. E. D.

89. *Is there nothing superfluous in Euclid's demonstration?*

It is unnecessary for him to repeat that part of Prop. IV. which shews that "the base AD = the base FC."

90. *Required to generalise Prop. XLVII?*

If upon the sides AB, AC of any △ ABC, there be described any ▱ˢ AG, AE; if the sides GF, ED when produced meet in H, and HA be joined; finally if upon the third side BC of the △, a ▱ CM be described, whose side BM is = and ‖ HA; then



$$\text{▱}\ BN = \text{▱}\ AE + \text{▱}\ AG.$$

Produce MB, NC to meet GF, ED or them produced in P and Q, and produce HA to meet MN in L, cutting also BC in K.

∵ the ▱ˢ AG, AP are on the same base and between the same ‖ˢ,

$$\text{▱}\ AP = \text{▱}\ AG.$$

Also, ∵ BM, BP each = AH,

$$\therefore BP = BM.$$

∴ ▱ˢ AP, BL are upon equal bases MB, BP, and are between the same ‖ˢ and are ∴ =.

But ▱ AG = ▱ AP,

$$\therefore \text{▱}\ AG = \text{▱}\ BL.$$

Similarly it may be shown that

$$▱\, AE = ▱\, CL.$$

Hence the whole ▱ BN = ▱ AE + ▱ AG.

Q. E. D.

This is not given as a perfect demonstration, for, to save prolixity, many things, which ought to have been proved, are assumed; such as L falling on MN not produced, &c. This general case is taken from Pappus.

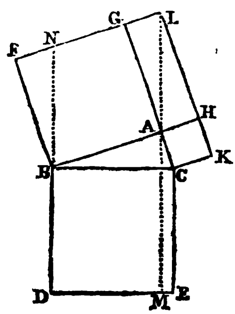
91. Hence is suggested another direct demonstration of Prop. XLVII., viz.

Produce FG, KH to meet in L. Join AL, and produce LA to meet DE in M. Also produce DB to meet FG in N.

$\because \angle NBC = ∟$,

and $\angle FBA = ∟$,

$\therefore \angle NBC = \angle FBA$.



Take away the common $\angle NBA$;

$\therefore \angle FBN = \angle ABC$.

Also $\angle F = \angle BAC$,

and $\because$ (Prop. XXXII.),

$\angle F + \angle FNB + \angle FBN =$

$\angle BAC + \angle ACB + \angle ABC$;

$\therefore \angle FNB = \angle ACB$.

Now, since in the △s BFN, ABC, the $\angle$s F and FNB = the $\angle$s BAC and ACB each to each, and the sides FB, AB, which subtend = $\angle$s, are = ; $\therefore$ by Prop. XXVI.,

the △s are = in all respects,

$\therefore BN = BC$, and $FN = AC$.

Again, since, as is proved in Euclid, BAH, CAG are $|$s, and BH, FL are $\|$, as also CG, KL,

$\therefore GH$ is a ▱.

Hence $GL = AH$ or AC.

But $FN = AC$;

$\therefore GL = FN$.

To each add NG.

$\therefore FG = NL$.

But $FG = BA$;

$\therefore NL = BA$,

and they are also $\parallel$.

$\therefore$ (Prop. XXXIII.),

AI is $=$ and $\parallel$ to BN, and AN is a ▱.

Now $\because$ the ▱^{s} AF, AN stand on the same base, and are between the same $\parallel^s$,

$AF = AN$.

Also, $\because$ the ▱^{s} AN, BM stand on equal bases BN, BD, and are between the same $\parallel^s$

$BM = AN$,

$\therefore \square AF =$ ▱ BM.

Similarly it may be shewn that the

$\square AK =$ ▱ CM,

$\therefore \square BE = \square AF + \square AK$,

&c. &c.

This proof of the Proposition, as well as all others we have seen, is decidedly inferior to Euclid's.

92. *Ought it not to have been a condition in Euclid's construction, that the □^{s} described on the sides of the △ should fall without the △?*

Amongst other reasons for such a condition is this, that if BE were described on the other side of BC, AL would not cut the $\square$ BE into two ▱^{s}. Its omission, therefore, renders the Proposition imperfect.

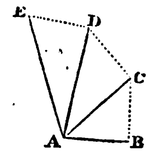
93. *What can be immediately deduced from this Proposition?*

(1.) That if upon each of the sides of a right-angled △ there be described a □, that upon either of the sides containing the ∟ is = to the difference of the □^{s} upon the other two sides.

(2.) Given the two sides of a right-angled $\triangle$ which contain the $\llcorner$, to find the *numerical value* of the side which subtends it; is done by extracting the square root of the sum of the $\square^s$ upon the given sides.

(3.) Given the side which subtends the $\llcorner$ and one of those which contain it, to find the other side *numerically;* this is done by extracting the square root of the difference of the $\square^s$ of the given sides.

(4.) To find geometrical values of 1, $\sqrt{2}$, $\sqrt{3}$, $\sqrt{4}$, $\sqrt{5}$, &c.



Let $AB = 1$, from B draw $BC \perp$ and $= AB$, and join AC; from C draw $BCD \perp BC$, and $= AB$. and join AD; from D draw $DE \perp AD$, and $= AB$, &c.

$$\because B \text{ is a } \llcorner, \text{and } AB = BC = 1,$$
$$\therefore AC^2 = BA^2 + CB^2 = 2BA^2 = 2;$$
$$\therefore AC = \sqrt{2}.$$

Again, for similar reasons,

$$AD^2 = AC^2 + CD^2 = 2 + 1 = 3;$$
$$\therefore AD = \sqrt{3}.$$

Again,

$$AE^2 = AD^2 + DE^2 = 3 + 1 = 4;$$
$$\therefore AE = \sqrt{4} = 2,$$
&c.

Hence the square roots of the natural numbers 1, 2, 3, 4, 5, &c. are represented geometrically by

AB, AC, AD, AE, &c.

Q. E. D.

Prop. XLVIII.

94. This is the converse of Prop. XLVII.

It must in this be recollected that, in the construction, BA is not *produced* (a mistake generally made by beginners), but that AD is *drawn at* $\llcorner^s$ to AC, and made equal to AB, &c.

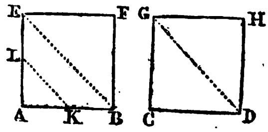
95. ***What does Euclid assume in the demonstration?***

That if two $\square^s$ are $=$, their sides are $=$ also.

First let us shew that

$\square^s$ described on $=$ straight lines are equal.

Let AF, CH be two $\square^s$ described on the $=$ $|^s$ AB, CD; $AF = CH$.



$$\text{Join } EB,\ GB. \text{ Then } \because \text{ in the } \angle^s\ EAB,\ GCD,$$
$$EA,\ AB = GC,\ CD \text{ each to each,}$$
$$\text{and included } \angle\ A = \text{included } \angle\ C, \text{ being both } ⊾^s,$$
$$\therefore \triangle\ EAB = \triangle\ GCD.$$

$$\text{But } \square\ AF = \text{two } \triangle^s\ EAB, \text{ and } \square\ CH = \text{two } \triangle^s\ GCD,$$
$$\text{and the doubles of equals are equal;}$$
$$\therefore \square\ AF = \square\ CH.$$

Q. E. D.

96. Again, *equal $\square^s$ are described upon $=$ $|^s$.*

Let the $\square^s$ AF, CH be $=$, AB shall $=$ CD.

If not, one of them must be $>$. Let AB be $>$ CD, then AE is also $>$ CG; from AB, AE cut off AK, AL each $=$ CD or CG, and join LK.

$$\because \text{ in the } \triangle^s\ LAK,\ GCD,$$
$$LA,\ AK = GC,\ CD \text{ each to each, and } \angle\ A = \angle\ C,$$
$$\therefore \triangle\ LAK = \triangle\ GCD.$$

$$\text{But } \triangle\ GCD = \tfrac{1}{2}CH = \tfrac{1}{2}AF = \text{the } \triangle\ AEB;$$
$$\therefore \triangle\ LAK = \triangle\ AEB, \text{ the } < \text{ the } > \text{ which is absurd;}$$
$$\therefore AB = CD.$$

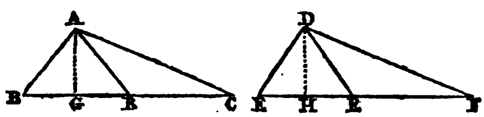
Q. E. D.

97. We shall here advert, although it may be somewhat out of place, to the subject of the identification of $\triangle^s$.

It has already been shewn that by the Propp. IV, VIII, and XXVI, Euclid has considered thirteen of the twenty combinations of the six sides and $\angle^s$ of a $\triangle$, combining

them three and three together. What are the remaining seven combinations?

If any two $\triangle^s$ have two sides of the one $=$ to two sides of the other, each to each, and one $\angle$ = one $\angle$, viz. those which are subtended by equal sides, then the $\triangle^s$ shall be equal in all respects or not, according as the $\triangle^s$ are or are not "of the same affection;" that is, according as the remaining $\angle^s$ subtended by equal sides are or are not both acute or both obtuse.



Let ABC, DEF be two $\triangle^s$ of which the sides AB, AC are equal to DE, DF, each to each, and let one $\angle\ C$ subtended by AB of the one equal one $\angle\ F$ subtended by DE, which is equal to AB, of the other; then if the $\triangle^s$ are of the same affection, that is, if the $\angle^s\ ABC$, DEF be both acute, or both obtuse, BC shall $= EF$, the $\angle\ BAC = \angle\ EDF$, and the $\angle\ ABC = \angle\ DEF$, and the $\triangle\ ABC = \triangle\ DEF$; but if they are not of the same affection, these $\angle^s$ and sides are not equal.

From A and D let fall upon CB, FE the $\perp^s\ AG$, DH respectively.

$\because$ in the $\triangle^s\ ACG$, DFH, there are two $\angle^s$ of the one $=$ to two $\angle^s$ of the other, each to each, viz. the $\llcorner^s$ and the $\angle^s\ C$ and F, and also the sides AC, DF, which subtend equal $\angle^s$ equal, $\therefore$ (by Prop. XXVI.),

$$AG = DH,\ GC = HF, \text{ and } \angle\ GAC = \angle\ HDF.$$

Again, since the $\triangle^s\ ABG$, AEH are right-angled at G and H, $\therefore$ (XLVII.),

$$AB^2 = AG^2 + GB^2,\ DE^2 = DH^2 + HE^2.$$

But $AB = DE$, and $AG = DH$;

$\therefore$ (see Art. 95),

$$AB^2 = DE^2, \text{ and } AG^2 = DH^2.$$

$$\therefore AG^2 + GB^2 = DH^2 + HE^2.$$

Take away the $=^s$ AG^2, DH^2,
and $GB^2 = HE^2$;

$\therefore$ (96),
$GB = HE$.

But $GC = HF$; $\therefore$ adding to, or subtracting from, the $=^s$ GC, HF, the $=^s$ GB, HE, according as CB, FE are $<$ or $>$ than CG, FH, the wholes or remainders CB, FE are $=$.

Again, in the $\triangle^s$ ACB, DFE, the
sides AC, CB = sides DF, FE each to each,
and $\angle^s$ included C and F are equal;

$\therefore$ $\angle^s$ subtended by equal sides are equal,
$\therefore \angle ABC = \angle DEF$,
and $\angle CAB = \angle FDE$.

Also $\triangle ABC = \triangle DEF$.

Now when CB, FE are $<$ than
CG, FH, G falls beyond B, and the exterior
$\angle^s$ ABC, DEF are $>$ than the interior $\angle^s$ AGB,
AHE, which are $\llcorner^s$,
$\therefore$ the $\angle^s$ ABC, DEF in this case are both obtuse.

Again, when CB, FE are $>$ than CG, FH, the
$\angle^s$ AGC, DHF are exterior, and B and E interior,
$\therefore$ $\angle^s$ AGC, DHF are $>$ than B and E.

But the former are $\llcorner^s$.
$\therefore$ B and E are $< \llcorner^s$, or are both acute.

Hence in both the cases demonstrated, the $\triangle^s$ ACB, DFE have the same affection.

It is evident from the construction that if one $\angle$ DEF be obtuse and the other acute, then CB is not equal to EF, &c.

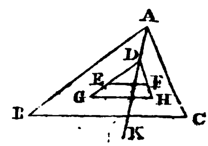
$\therefore$ if two $\triangle^s$ have two sides of the one, &c.

Q. E. D.

This case applies to six of tne seven remaining combinations of the sides and $\angle^s$ of a $\triangle$.

It remains only to consider that combination in which *three* ∠ *in one* △ *equal the three* ∠ˢ *of the other* △ *each to each.*

Let ABC be any △. Within it take any point D, and from D draw $DE \parallel AB$, and $DF \parallel A$, and from any other points within the △ and in DE, as E, G, draw |ˢ ∥ BC, meeting DH in F and H. Join AD and produce it to K.



$$\because ED \text{ is } \parallel AB, \text{ and } AD \text{ meets them,}$$
$$\therefore \angle KDE = \angle KAB,$$
$$\text{similarly } \angle KDF = \angle KAC;$$
$$\therefore \text{whole } \angle EDF = \text{whole } \angle BAC.$$

Similarly by joining CF and BE, it may be shewn that
$$\angle DEF = \angle B, \text{ and } \angle DFE = \angle C.$$

∴ the three ∠ˢ of the △ DEF = the three ∠ˢ of the △ ABC each to each.

In like manner innumerable other △ˢ such as DGH may be described equiangular with ABC.

Hence this case is perfectly *indeterminate.*

BOOK II.

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98. *What is the subject of Book II?*

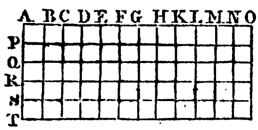
It treats of the surfaces of □s and ▭s.

99. *What is a Rectangle?*

It is a right-angled parallelogram, or the same as an oblong (see Definition XXXI., Book I.)

PROP. I.

100. *Required to shew that by this Proposition may be established the theory of multiplication geometrically.*



Let $AB = 1$ (1 inch, 1 foot, 1 lb. or any other unit).

Also produce AB, and take $BC = AB$, $CD = AB$, &c., to a terms, then $AO = AB + BC + \text{\&c.} = 1 + 1 + 1 + \text{\&c.} = a$.

Again, at A in the | AO, draw $AP \perp AO$, and make $AP = AB = 1$, produce AP, make $PQ = AP$, $QR = AP$, &c., and let b of these equals be taken. Then

$$AT = 1 + 1 + 1 + 1 \ldots b \text{ terms} = b.$$

$\therefore AO$ denotes the number a, and AT the number b.

Now ▭ TO (by Prop. I, Book II) = fonr of the ▭s contained by AT and by each of the parts of the divided line AO; but these parts are all equal, and are in number a.

$$\therefore \text{▭ } TO = a \text{ times ▭ } TB;$$

$$\therefore \text{▭ } TO = \text{▭ } TB \times a.$$

Again ▭ TB = the sum of the ▭s contained by the un-

divided line AB, and by each b of the equal parts of the divided line AT

$$\therefore ▭\ TB = \square\ PB \times b;$$
$$\therefore ▭\ TO = (\square\ PB \times b) \times a.$$

Let that □ whose side is unit, be itself the unit of surface (as a square inch, a square foot, or any other denomination), then

$$\square\ PB = 1.$$

Hence

$$▭\ TO = (1 \times b) \times a = b \times a,$$

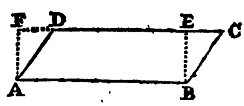
that is, *the surface of a ▭ whose sides are a and b, is measured by the product arising from the multiplication of one side by the other.*

Hence the reason for denoting a ▭ by the product of the two sides which contain it. Thus, in fig. of Prop. I, it is sometimes written

$$BC \times BG,$$

meaning thereby the ▭ GC.

101. *To measure the area or surface of any* ▱



Let $ABCD$ be any ▱, it is required to find its area. From B and A draw BE, $AF \perp CD$, produced when necessary.

Then the ▭ BF and ▱ AC being upon the same base AB, and between the same $\|^s$ AB, CF,

▱ AC = ▭ BF (Prop. XXXV., Book I.)

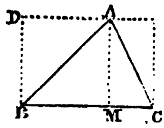
But (100)

$$▭\ BF = AB \times AF,$$
$$\therefore ▱\ AC = AB \times AF;$$

that is, the surface of any ▱ = the product of its base by the $\perp$ drawn from its base to the opposite side.

102. Hence *the area of a □ = side × side = (side)²*.

103. *To find the area of a* $\triangle$.



Let ABC be any $\triangle$, upon the same base BC, let CD be a ▭, and from A draw AM $\perp$ BC, or BC produced.

Now the ▭ CD and $\triangle$ ABC, being upon the same base and between the same $\|^{s}$,

$$\triangle ABC = \tfrac{1}{2} \square CD.$$

$$\text{But } \square CD = CB \times BD \text{ (by 100)},$$

and $BD = AM$, because they are opposite sides of a ▭.

$$\therefore \square CD = CB \times AM.$$

$$\therefore \triangle ABC = \tfrac{1}{2} CB \times AM;$$

that is, *the area of any* $\triangle$ *is measured by half the product of its base and the* $\perp$ *let fall from the vertex upon the base.*

104. *To find the area of any rectilineal figure.*

First divide it into $\triangle^{s}$, and then find the area of each $\triangle$ separately by (102). The sum of all these areas will be the area required.

These few rules contain the entire theory of Land-surveying, the Mensuration of Ceilings, Floors, &c. &c.

105. *To determine Prop. I. Algebraically.*

$$\begin{aligned} BC \times A &= (BD + DE + EC) \times A \\ &= BD \times A + DE \times A + EC \times A. \end{aligned}$$

Q. E. D.

Hence it appears that Prop. I. *demonstrates Geometrically the rule of Algebra, which inculcates that the product of two quantities is equal to the sum of the products of one of them multiplied by the several parts.*

PROP. II.

106. *Demonstrate this Proposition by the rules of Algebra.*

$$AC + CB = AB,$$
$$\therefore (AC + CB) \times AB = AB \times AB.$$
$$\therefore AC \times AB + CB \times AB = AB^2 \text{ (102)}.$$

Q. E. D.

PROP. III.

107. *Demonstrate this Algebraically.*

Since

$$AB = AC + CB,$$
$$\therefore AB \times BC = (AC + CB) \times BC$$
$$= AC \times BC + BC^2.$$

Q. E. D.

PROP. IV.

108. *To prove this Algebraically.*

Since

$$AB = AC + CB,$$
$$\therefore AB^2 = (AC + CB)^2$$
$$= AC^2 + 2AC \times CB + CB^2,$$

by the Binomial Theorem, or by actual Multiplication.

Hence it appears that Euclid's IV. Prop., Book II., is a Geometrical proof of the Binomial Theorem, for that case in which the index is 2.

PROP. V.

109. *Prove this Algebraically.*

Since

$$AD = AC + CD,$$
$$\text{and } DB = BC - CD = AC - CD,$$
$$\therefore AD \times DB = (AC + CD)(AC - CD)$$
$$= AC^2 - CD^2.$$
$$\therefore AD \times DB + CD^2 = AC^2.$$

Q. E. D.

Prop. V. Cor.

110. *"The difference of the* $\square^s$ *of two unequal lines* AC, CD *is equal to the* $\square$ *contained by their sum and difference;"* required to shew this more explicitly than Euclid has done.

Taking Euclid's construction

$$AC^2 = CB^2 = CF,$$
$$\text{and } CD^2 = LG,$$

$\therefore$ the difference between AC^2 and CD^2 is the difference between CF and LG. But this latter difference = the gnomon CMG.

$$\text{But of the gnomon, } DF = AL,$$
$$\therefore \text{ gnomon} = AH.$$
$$\text{But } AH = AD \times DB,$$
$$\therefore AC^2 - CD^2 = AD \times DB.$$
$$\text{But } AD = AC + CD,$$
$$\text{and } DB = CB - CD \text{ and } CB = AC,$$
$$\therefore DB = AC - CD.$$

Hence

$$AC^2 - CD^2 = (AC + CD)(AC - CD).$$

Q. E. D.

This Cor. forms an Algebraical theorem of the utmost utility.

Otherwise Algebraically.

By Multiplication,

$$(AC + CD)(AC - CD) = AC^2 + AC \times CD - AC \times CD - CD^2$$
$$= AC^2 - CD^2.$$

Q. E. D.

Prop. VI.

111. *Prove this Algebraically.*

$$AD \times DB + CB^2 = (CD + AC)(CD - CB) + CB^2.$$
$$\text{But } CA = CB,$$

$$\therefore AD \times DB + CB^2 = (CD + BC)(CD - CB) + CB^2$$
$$= CD^2 - CB^2 + CB^2$$
$$= CD^2.$$

Q. E. D.

Prop. VII.

112. *Prove it Algebraically.*

$$AB^2 + BC^2 = AB^2 + (AB - AC)^2$$
$$= AB^2 + AB^2 - 2AB \times AC + AC^2$$
$$= 2AB^2 - 2AB \times AC + AC^2$$
$$= 2AB(AB - AC) + AC^2$$
$$= 2AB \times BC + AC^2.$$

Q. E. D.

Prop. VIII.

113. *Prove this Algebraically.*

$$AD^2 = (AB + BC)^2 = AB^2 + 2AB \times BC + BC^2.$$

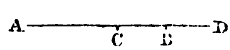
But

$$BC^2 = (AB - AC)^2$$
$$= AB^2 - 2AB \times AC + AC^2,$$

$$\therefore AD^2 = 2AB^2 + 2AB \times BC - 2AB \times AC + AC^2$$
$$= 2AB(AB - AC) + 2AB \times BC + AC^2$$
$$= 2AB \times BC + 2AB \times BC + AC^2$$
$$= 4AB \times BC + AC^2.$$

Q. E. D.

114. An easier Geometrical Demonstration.



Since AD is divided into two parts in B, $\therefore$ by Prop. IV.,

$$AD^2 = AB^2 + BD^2 + 2AB \times BD.$$

Again, since AB is divided in C into two parts, by Prop. VII.,

$$AB^2 + BC^2 (= BD^2) = 2AB \times BC + AC^2,$$

$$\therefore AD^2 = 2AB \times BC + AC^2 + 2AB \times BD.$$

$$\text{But } BD = BC,$$
$$\therefore AD^2 = 4AB \times BC + AC^2.$$

Q. E. D.

Prop. IX.

115. *Prove this Algebraically.*

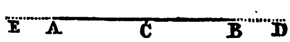
$$\begin{aligned} AD + DB^2 &= (AC + CD)^2 + (BC - CD)^2 \\ &= AC^2 + 2AC \times CD + CD^2 + BC^2 \\ &\quad - 2BC \times CD + CD^2. \end{aligned}$$

$$\text{But } CB = AC,$$
$$\therefore AD^2 + DB^2 = 2AC^2 + 2CD^2.$$

Q. E. D.

Prop. X.

116. An easier Demonstration.



Let B be bisected in C, and produced to any point D;

$$AD^2 + BD^2 = 2AC^2 + 2CD^2.$$

Produce CA to E, making $AE = BD$.

$\because$ DE is divided into two equal parts in C, and into two unequal parts in A,

$$\therefore DA^2 + AE^2 = 2DC^2 + 2CA^2 \text{ (Prop. IX.)},$$
$$\text{But } AE = BD, \therefore BD^2 = AE^2 \text{ (95)},$$
$$\therefore AD^2 + BD^2 = 2AC^2 + 2CD^2.$$

Q. E. D.

117. *Required the Algebraical demonstration.*

$$\begin{aligned} AD^2 + BD^2 &= (AC + CD)^2 + (CD - AC)^2 \\ &= AC^2 + 2AC \times CD + CD^2 + CD^2 \\ &\quad - 2AC \times CD + AC^2 \\ &= 2AC^2 + 2CD^2. \end{aligned}$$

Q. E. D.

Prop. XI.

118. *Demonstrate this by Algebra.*

Let $AB = a$, $AH = x$, then $HB = a - x$,

$\therefore$ by the question

$$a\ (a - x) = x^2.$$

$$\therefore\ x^2 + ax = a^2.$$

$$\therefore\ x = \sqrt{\left(a^2 + \frac{a^2}{4}\right)} - \frac{a}{2} = \frac{a}{2}\ (\sqrt{5} - 1)$$

which determines the point H.

Q. E. I.

The above expression for x has two values, $\because$ $\sqrt{5}$ is positive or negative; what results from its having two values?

That there are two solutions of the question when considered generally, that is, analytically. The first solution, in supposing $\sqrt{5}$ positive, corresponds to Euclid's construction. The second, in making $\sqrt{5}$ negative, gives the point H in BA produced, at a distance from A equal to

$$\frac{\sqrt{5} + 1}{2} \times AB.$$

Prop. XIII.

119. *Aliter proof of Case* 2.

Instead of making use of Prop. XII. Book II. as Euclid does, the second case may be proved by Prop. VII., in the same words exactly as the first case, with the exception of the commencement, viz.

$\because$ the | BD (Euclid's second fig.) is divided into two parts in the point D; after which

"$\therefore$ (7, Book II.) $CB^2 + BD^2 = 2CB \times BD + CD^2$,

to each add AD^2.

$$\therefore CB^2 + BD^2 + AD^2 = 2CB \times BD + CD^2 + AD^2.$$

But

$AB^2 = BD^2 + AD^2 \because \angle D$ is a ∟;

and $AC^2 = CD^2 + AD^2$,

$$\therefore CB^2 + AB^2 = AC^2 + 2CB \times BD, \text{ \&c.''}$$

which is Euclid verbatim.

BOOK III.

120. *What is the subject of this Book?*

It treats of the properties of Circles.

Definition IV.

121. *What is the "distance" between two points?*

In language,

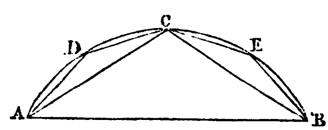
It is the length of *a line*, whose extremities are these points.

But in Geometry,

It is the length of the *straight line*, whose extremities are these points.

122. *What is the shortest line that can be drawn from one point to another?*

Let A, B be the points, AB a straight line, and ACB a curved line joining them. AB is shorter than ACB.



In ACB take any point C, join AC, BC. Then (XX. B. I.)

$$AC + CB > AB.$$

Again in AC take any point D, and in BC take any point E, &c. Then

$$AD + DC > AC \text{ and } CE + EB > CB.$$

$$\therefore AD + DC + CE + EB > AC + BC \text{ and much} > AB.$$

Hence proceeding we show that every exterior line $ADCEB$ is > than the next interior viz. ACB, and consequently it is evident that the curve is > AB.

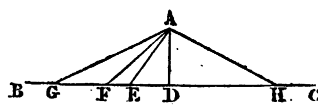
In the above figure the curve is wholly concave towards

AB, but if it be partly convex and partly concave; partly rectilinear as well as curvilinear; partly above and partly below the line *AB*, the same method will apply, with very slight modifications.

From which then it follows that *the shortest line that can be drawn from one point to another is the straight line.*

The above is the method of Legendre.

123. *What is the shortest line that can be drawn from a given point to a given | ?*



Let A be the given point. BC the given | . From A let fall a $\perp$ AD upon BC, take any points E, F, G, H, and join on either side of D, AE, AF, AG, AH, &c.

Then because ADE is a $\llcorner$, AED is less than a $\llcorner$, and the greater $\angle$ is subtended by the greater side,

$$\therefore AE \text{ is } > AD.$$

Again, since

$$\angle AEF \text{ is } > ADE > \llcorner,$$
$$\therefore \angle AFE \text{ is } < \text{ than a } \llcorner.$$
$$\therefore \angle AEF \text{ is } > \text{ the } \angle AFE.$$
$$\therefore \text{(XIX., Book I.)}, AF \text{ is } > AE, \text{ and much } > AD.$$

Hence by similarly proceeding it follows that *of all the straight lines which can be drawn from the point A to the | BC, the $\perp$ AD is the least, and that which is nearer to the $\perp$ is always less than that which is more remote.*

124. *Which of the points of a given | is nearest to a given point?*

Let A be the given point, and BC the given | . Then from the preceding article it appears that D is the point required.

Hence is suggested the following definition.

125. *How is defined the distance of a given | from a given point?*

It is the distance of the nearest point of the given | from the given point.

These preliminaries being explained, it becomes easy to perceive why Euclid has defined straight lines which are equally distant from the centre of a ⊙ to be those to which ⊥ˢ drawn to them from the centre are =.

Definition XI.

126. *What is the ambiguity of this definition?*

The definition involves Prop. XXI. Book III., for it supposes that all the ∠ˢ of either segment are = to one another.

Prop. I.

127. *Does Euclid's demonstration apply to all positions of the point G?*

When G is taken in the | CE it does not, because then the ∠ FDB is not > GDB.

This case, however, is almost self-evident, because if G, the supposed centre, be any point in CE other than F, GC would be = GE; but $FC = FE$, from which an absurdity immediately follows.

If G be in CE it is evidently the middle point of CE; that is, it is F.

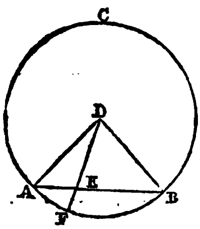
Prop. I. Cor.

128. Hence it is also clear that *if two straight lines, such as CE, be drawn, the centre of the circle, being in each of them, must be their point of intersection; which gives a practical method of finding the centre of a* ⊙.

Prop. II.

129. *Give Commandine's direct demonstration.*

In AB take any point E, join DE, and let it be produced, if necessary, to meet the $\odot^{ce}$ in F.



$\because \angle DEB$ is exterior it is $> \angle DAE$. But $\because DA = DB$,
$\therefore \angle DAE = \angle DBA$.
$\therefore \angle DEB$ is $>$ than $\angle DBE$.
$\therefore$ (18, Book I.), DE is $<$ than DB.
But $DB = DF$,
$\therefore DE$ is $<$ than DF.

$\therefore$ the point E falls within the $\odot$; and the same may be shown in the same manner of every point in AB. $\therefore$ &c.

Q. E. D.

130. *What is the supposed defect of Commandine's proof?*

Some imagine it faulty in assuming the axiom that $\because DE$ is shorter than $DF \therefore E$ falls within the $\odot$. This seems, however, mere cavil.

Prop. III.

131. *Why is it a condition in the enunciation that the line bisected shall* not *pass through the centre?*

Because, if both lines pass through the centre they must bisect each other (15, Book I.), but may be inclined to one another at any $\angle$ whatever. This applies also to Prop. IV.

Prop. VII.

132. *Euclid has demonstrated the Proposition for those only which are on the same side of the diameter AD; is it also true, when they are on different sides?*

Let FK be more remote from the diameter than FG. Make $FH = FG$, as in Euclid, then $\because \angle HED = \angle GED$, FH is equally remote with FG. Hence, as in Euclid, FH is less than FK, &c.

Prop. VIII.

133. *Which is the longest | that can be drawn from any point without the ⊙ce of a ⊙ to that portion which is convex towards the point?*

That which, if produced, does not cut the ⊙.

Prop. IX.

134. *Prove by means of Prop. VII. and VIII. that there is no point, whether within or without a ⊙, except the centre, from which three = |s can be drawn to the ⊙ce.*

If the point be within the ⊙ce, it has been shown by Euclid (Prop. IX.), that it must be centre.

If without, as D (see fig. Prop. VIII.) let, if possible, the three |s DB, DK, DN be =. Find M the centre of the ⊙, join DM and produce it so as to meet the ⊙ce in G and D. Then (by VIII.) since M is the centre and D a point in the | DM, which is not the centre, $\therefore$ there cannot be more than two = |s drawn from D to the ⊙ce. $\therefore$ DK, DB, DN are not all =, which is absurd. $\therefore$ D is the centre, &c.

Q. E. D.

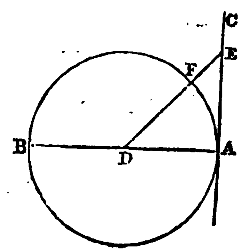
Prop. XIV.

135. *What does Euclid assume in the demonstration?*

"$\because AE = EC, \therefore AE^2 = EC^2$."
This requires proof (see 95).

Prop. XVI, and Cor.

136. An easier Demonstration.



If AC be $\perp$ to the diameter AB at its extremity A, AC touches the $\odot$ in A.

For in AC take any point E, join DE, intersecting the $\odot^{ce}$ in F.

Then, $\because DAE$ is a $\llcorner$ the $\angle^s$ ADE, DEA are together $=$ a $\llcorner$.

$\therefore$ the $\angle$ AED is $<$ than $\llcorner$.

$\therefore$ the $\angle$ AED is $<$ the $\angle$ DAE.

$\therefore$ side DE is $>$ side DA.

But $DA = DF$.

$\therefore$ DE is $>$ DF.

$\therefore$ the point E falls without the $\odot^{ce}$.

In the same way it may be shewn that any other point in AC or AC produced falls without the $\odot^{ce}$.

$\therefore$ by Definition II., Book III.,

AC touches the $\odot$ in A.

Q. E. D.

137. Prove the converse of Prop. XVI., viz. that

If a | touch a $\odot$ at any point, it is at $\llcorner^s$ to the diameter which passes through the point.

Let AE (see Euc. fig.) touch the $\odot$ in A. Take D the centre, and join AD. The $\angle$ DAE is a $\llcorner$.

If DAE be not a $\llcorner$ either DAE or the $\angle$ which DA makes with EA produced, is an acute $\angle$.

Let DAE be an acute $\angle$ then (Prop. XVI., Book III.) AE must cut the $\odot^{ce}$. But it does not, because it touches it; which is absurd. $\therefore$ the $\angle$ DAE is not acute.

In the same way, if it be supposed that $\angle$ DAE is obtuse, or the $\angle$ made by EA produced and DA acute, it may be shewn that the latter $\angle$ is not acute, and $\therefore$ that the $\angle$ DAE is not obtuse. $\therefore$ the $\angle$ DAE is neither acute nor obtuse; it is $\therefore$ a $\llcorner$. Q. E. D.

Compare this with Prop. XVIII.

138. *Prove that "there can be but one* $|$ *which touches the* $\odot$ *in the same point."*

If possible, let AE, AF (see Euc.) both touch the $\odot$ AHB in the same point A. Take D the centre and join DA. Then, $\because$ EA touches the $\odot$ in A, the $\angle$ DAE is a $\llcorner$ (137).

Also, $\because$ FA touches the $\odot$ in A, the $\angle$ DAF is a $\llcorner$.

$$\therefore \angle DAE = \angle DAF.$$

But they are unequal, which is absurd.

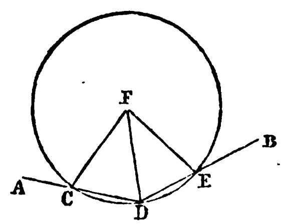
$\therefore$ AF and AE cannot both touch the $\odot$ in A. and in the same way it may be shewn that no $|$ but AE can touch the $\odot$ in the point A.

$\therefore$ &c.

Q. E. D.

139. *Prove that a* $|$ *cannot cut a* $\odot$ *in more than two points.*

If possible, let the $|$ AB cut the $\odot$ CDE in three points C, D, E. Take F the centre of the $\odot$ and join FC, FD, FE.



$\because$ F is the centre,

$$\therefore FE = FD = FC;$$

$$\therefore \angle FED = \angle FDE,$$

and $\angle FDC = \angle FCD$.

$$\therefore \angle FCD + \angle FED = \angle FDC + \angle FDE.$$

But $\because$ FD meets the $|$ AB in D, $\therefore$

$$\angle FDC + \angle FDE = \text{two } \llcorner^s.$$

$$\therefore \angle FCD + \angle FED = \text{two } \llcorner^s.$$

But any two ∠s of the △ $FCDE$ are together < two ∟s.

∴ ∠ FCD + ∠ FED is also less than two ∟s,
which is absurd.

∴ AB does not cut the ⊙ in three points.

Q. E. D.

Prop. XVII.

140. In Euclid's construction it is evident that both AB and DF touch the ⊙, but at different points.

141. *From a given point without a ⊙ can there be drawn more than one tangent?*

If AB (see Euc.) were drawn on the other side of EF, A and D would also fall on the other side, which would give the tangent FD another situation. Hence from a point without a ⊙ two tangents may be drawn on different sides of the diameter which passes through the point.

By an *Ex Absurdo* it would be easy to shew that not more than two can thus be drawn.

Prop. XX.

142. *In this Proposition Euclid has supposed two cases, viz. those in which E is first within the ∠ BAC; and secondly, without the ∠ BAC; but how is to be demonstrated that case in which E is neither* within *nor* without *the ∠ BAC, viz. that in which E is in either AB or AC?*

If E be in either side which contains the ∠ at the ⊙ce, as at E in the | AF (see Euclid's fig.); then ∠ FEC = ∠ EAC + ∠ ECA = 2 ∠ EAC ∵ E is the centre of the ⊙ &c. &c.

Prop. XXIII.

143. *Why does DB cut the segment ACB in C?*

No reason is assigned.

It would have been better thus. In the interior segment ACB take any point C, join BC and produce it to the exterior segment in D, &c. &c.

PROP. XXVI.

144. *Hence, in the same* $\odot$ = $\angle^s$ *stand upon* = $\odot^{ces}$.

PROP. XXIX.

145. *Hence, in the same* $\odot$ = $\odot^{ces}$ *are subtended by* = $|^s$.

PROP. XXXI.

146. *Required from this Proposition to deduce an easy practical method of drawing a* $\perp$ *to a given* | *from a given point in it? (aliter method of Prop.* XI., Book I.)

Let A be any point in the given | BF (see fig. Euc.), take any point E which is not in the | BF, join AE, and with centre E and distance EA describe a $\odot$ BAC, cutting the | in B, A, or touching it in A.

If the $\odot$ touch the | in A, then AE is $\perp$ to BF (16, Book III.)

Q. E. I.

But if it cut in B and A, then join BE and produce it to the circumference in C and join AC. AC shall be $\perp$ BC in the point A.

$\because$ BC passes through the centre, BAE is a semi-circle. But the $\angle$ in a semi-circle is a $\llcorner$.

$\therefore$ $\angle$ BAC is a $\llcorner$, or CA is drawn from A $\perp$ BF.

Q. E. I.

PROP. XXXI. COR.

147. This Corollary is deducible also from Prop. XXXII., Book I.

For the $\angle^s$ of a $\triangle$ being A, B, C, if

$$A + B = C,$$
$$\therefore A + B + C = 2C \text{ (ax. II.)}$$

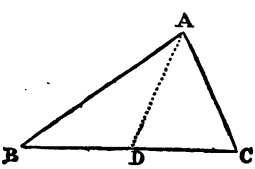
But by XXXII., Book I.,

$$A + B + C = \text{two } \llcorner^s,$$
$$\therefore 2C = \text{two } \llcorner^s \text{ (ax. I.)}$$
$$\therefore C = \text{a } \llcorner \text{ (ax. VII.)}$$

Hence also $A + B = \text{a } \llcorner$.

148. Prove the converse of Prop. XXXI.; viz. that

If a $\triangle$ be right-angled, the $\llcorner$ is the $\angle$ of that semicircle whose diameter is the side of the $\triangle$ which is opposite to the $\llcorner$.



Let ABC be a $\triangle$ right-angled at A; it is the $\angle$ of that semi-circle whose diameter is BC.

At A in AC make $\angle\ CAD = \angle\ C$, then $\because \angle\ BAC = \llcorner$,

$\therefore C + B = \llcorner$ (XXXII., Book I., or Cor. XXXI., Book III.)

$$\text{But } \angle\ CAD + \angle\ DAB = \llcorner,$$
$$\therefore C + B = \angle\ CAD + \angle\ DAB.$$
$$\text{But } \angle\ C = \angle\ CAD,$$
$$\therefore \angle\ B = \angle\ DAB.$$

Hence

$$DA = DB.$$
$$\text{But } \because \angle\ DAC = C,$$
$$\therefore DC = DA.$$
$$\therefore DA = DB = DC;$$

and if from the centre D at the distance DB a $\odot$ be described, it shall pass through B, A, C.

Again, $\because$ from the point D there can be more than two equal $|^s$ drawn to the $\odot^{ce}$, viz. DB, DA, DC,

$$\therefore D \text{ is the centre of the } \odot.$$

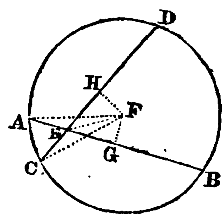
Again, ∵ D is the centre, and BC passes through it, and is terminated both ways by the ⊙^{ce} in B and C, BC is the diameter of the ⊙; and ∵ BC is a diameter of the ⊙, the segment BAC is a semi-circle; ∴ the ∟ BAC is the ∠ of the semi-circle BAC, whose diameter is BC.

Q. E. D.

This converse is of great use in many enquiries. It is assumed in Newton's Lemma XI. of the Principia, and in many other places.

Prop. XXXV.

149. *Required to prove the Proposition for all cases by one construction.*



Let AB, CD be any two |ˢ cutting one another in E.

$$AE \times EB = CE \times ED.$$

Find F the centre of the ⊙.

Then, if the centre of the ⊙ be E,

$$AE \times EB = CE \times ED,$$

∵ AE, EB, CE, ED are all equal.

Again, if E be not the centre, join EF.

Then, if only one of the lines AB, CD, as CD pass through the centre F, from F let fall $FG \perp AB$, and join AF. The | $CEFD$ will be divided into two = parts in F, and two unequal parts in E; ∴ (Prop. V., Book II.),

$$CE \times ED + EF^2 = CF^2 = AF^2.$$

Again, ∵ AB is divided into two = parts in G, and two unequal parts in E, ∴

$$AE \times EB + EG^2 = AG^2.$$

To these =ˢ add FG^2,

$$\therefore AE \times EB + EG^2 + FG^2 = AG^2 + FG^2.$$

But $EG^2 + FG^2 = EF^2$, and $AG^2 + FG^2 = AF^2$.

$$\therefore AE \times EB + EF^2 = AF^2.$$

$$\text{But } CE \times ED + EF^2 = AF^2,$$
$$\therefore AE \times EB + EF^2 = CE \times ED + EF^2.$$
$$\therefore AE \times EB = CE \times ED.$$

Lastly, if neither AB nor CD pass through F, let fall $FH \perp CD$, and join CF.

Then, as before, since AB, CD are bisected in H, and cut into two unequal parts in the (•) E, $\therefore$

$$AE \times EB + EG^2 = AG^2,$$
$$\text{and } CE \times ED + EH^2 = CH^2.$$

To the first =^s add FG^2, and to the other =^s add FH^2, then,

$$AE \times EB + EG^2 + FG^2 = AG^2 + FG^2,$$
$$\text{and } CE \times ED + EH^2 + FH^2 = CH^2 + FH^2.$$

$$\text{But } EG^2 + FG^2 = EF^2,\ AG^2 + GF^2 = AF^2,$$
$$EH^2 + FH^2 = EF^2, \text{ and } CH^2 + FH^2 = CF^2, \text{ which} = AF^2.$$

$$\therefore AE \times EB + EF^2 = AF^2,$$
$$\text{and } CE \times ED + EF^2 = AF^2,$$
$$\therefore AE \times EB + EF^2 = CE \times ED + EF^2.$$
$$\therefore AE \times EB = CE \times ED.$$

∴ if two |^s, &c.

Q. E. D.

150. Hence, *if a | be drawn from the ⊙^ce of a ⊙ ⊥ diameter, the □ of the ⊥ shall be = to the ▭ contained by the segments of the diameter.*

For if the ⊥ be produced to meet the ⊙^ce on the other side of the diameter, the part produced = the ⊥ (Prop. III., Book III.)

151. Required to prove the converse of Prop. XXXV., viz. that

If two |^s intersect so that the ▭ contained by the segments of the one = ▭ contained by the segments of the other, the extremities of the |^s are in the ⊙^ce of a ⊙.

Let AB, CD (see fig. to 149) cut one another in E so that

$$AE \times EB = CE \times ED,$$

then A, B, C, D lie in the ⊙ce of a ⊙.

Bisect AB in G and CD in H; then if E be the (·) of bisection of both lines, it is evidently the centre of a ⊙ passing through A, B, C, D.

But if not, from H draw $HF \perp CD$, and from G draw $GF \perp AB$, then if H, G be joined it may easily be shown that $\angle FGH + \angle FHG$ is $<$ two ∟s, and ∴ that GF, HF will meet. Let them meet in F, and join FE, FA, FC.

Then if F fall upon either line as CD, the | $CEFD$ will be bisected in F and divided into two unequal parts in E.

∴ (Prop. V., Book II.),

$$CE \times ED + EF^2 = CF^2.$$

Also AB is bisected in G and divided into two unequal parts in E; ∴

$$AE \times EB + EG^2 = AG^2;$$

to each add FG^2,

$$\therefore AE \times EB + EG^2 + FG^2 = AG^2 + FG^2,$$

or (Prop. XLVII., Book I.),

$$AE \times EB + EF^2 = AF^2.$$

But by hypothesis,

$$AE \times EB = CE \times ED,$$

$$\therefore CE \times ED + EF^2 = AF^2.$$

$$\text{But } CE \times ED + EF^2 = CF^2,$$

$$\therefore AF^2 = CF^2.$$

$$\therefore AF = CF.$$

Now if FD be joined, ∵ $CH = HD$, and FH is common to the △s CHF, HDF, and the included ∠s are ∟s, ∴

$$FD = FC.$$

In the same way it may be shown that

$$FB = FA,$$

$\therefore FA = FB = BC = FD$, and A, B, C, D lie in a $\odot^{ce}$ whose centre is F.

Again, if F fall neither upon AB nor CD, then, $\because$ CD is bisected in H and cut into two unequal parts in E,

$$\therefore CE \times ED + EH^2 = CH^2;$$

add to each FH^2,

$$\therefore CE \times ED + EH^2 + FH^2 = CH^2 + FH^2.$$

$\therefore$ (Prop. XLVII., Book I.),

$$CE \times ED + EF^2 = FC^2.$$

But, as before,

$$AE \times ED + EF^2 = FA^2;$$

and by hypothesis,

$$CE \times ED = AE \times ED,$$

$$\therefore FA^2 = FC^2.$$

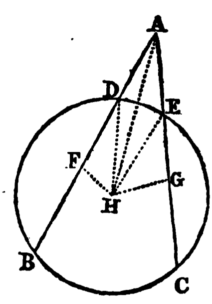
$\therefore FA = FC$, &c., as before.

Q. E. D.

Prop. XXXVI. Cor.

152. Required to prove the converse of this corollary, viz. that

If two $|^s$ *meet at a* $(\cdot)$ *and are so divided that the* $\square$ *contained by the whole, and the part nearest the vertex of the one, is = to the* $\square$ *contained by the whole and the part nearest the vertex of the other, then the* $(\cdot)^s$ *of section and the extremities of the* $|^s$ *which are not in the vertex are in the* $\odot^{ce}$ *of a* $\odot$.



Let AB, AC, any two $|^s$ meeting in A, be so divided in D and E that

$$AB \times AD = AC \times AE; \text{ then}$$

B, D, E, C shall be in the $\odot^{ce}$ of a $\odot$.

Bisect BD in F and EC in G; from F draw $FH \perp AB$, and from G draw $GH \perp AC$. If FG be joined it may easily be shown that the $\angle\, HFG +$

$\angle HGF$ is $<$ than two $\llcorner^s$, and $\therefore$ that FH, GH will meet. Let them meet in H. Join AH, DH, EH.

$\because$ BD is bisected in F and produced to the $(\cdot)$ A,

$\therefore$ (6, Book III.),

$$BA \times AD + FD^2 = FA^2.$$

To each add FH^2, then

$$BA \times AD + FD^2 + FH^2 = FA^2 + FH^2.$$

$\therefore$ (Prop. XLVII., Book I.),

$$BA \times AD + HD^2 = HA^2.$$

Similarly

$$CA \times AE + HE^2 = HA^2.$$

$$\therefore BA \times AD + HD^2 = CA \times AE + HE^2.$$

But $BA \times AD = CA \times AE$ (hypothesis),

$$\therefore HD^2 = HE^2.$$

$$\therefore HD = HE.$$

Now if HB, HC be joined, it may be shown by Prop. IV., Book I. that

$$HB = HD, \text{ and } HC = HE,$$

$\therefore$ HB, HC, HD, HE are all equal.

$\therefore$ B, C, D, E are in the $\odot^{ce}$ of which the centre is H.

$\therefore$ If, &c.

Q. E. D.

153. Prove that

Two $|^s$ *which touch a* $\odot$, *and are drawn from the same* $(\cdot)$, *are* $=$ *to one another.*

From the $(\cdot)$ where they meet draw a $|$, cutting the $\odot^{ce}$ in two $(\cdot)^s$; then (Prop. XXXVI., Book III.) the $\square$ of each tangent is $=$ to the $\square$ contained by the whole line and the segment of it without the $\odot$.

$\therefore$ the $\square^s$ of the tangents are equal,

and $\therefore$ the tangents themselves are equal.

Q. E. D.

BOOK IV.

154. *What is the subject of this Book?*

The inscription of rectilinear figures in ⊙ˢ, and their circumscription about ⊙ˢ; as also the inscription and circumscription of ⊙ˢ in and about rectilinear figures.

155. *Do any propositions relate to the inscription of rectilinear figures in, or their circumscription about, rectilinear figures?*

Although Definitions I. and II. define such inscription and circumscription, Euclid has not thought fit to give any examples, confining his views to ⊙ˢ and rectilinear figures only.

Prop. I.

156. *In the enunciation what is meant by "given ⊙" and by "given | "?*

That the ⊙ is given in position and magnitude; but that the line is given in magnitude only.

157. *Why is it a condition that the given | is not greater than the diameter of the given ⊙?*

∵ then the problem would be impossible, the diameter being the longest line in a ⊙

Prop. II.

158. *In the construction is it indifferent on which side of the tangent GH the |ˢ AC, AB are drawn?*

It ought to have been said "at the point *A* in the | *AH*," and *on the same side of it with the* ⊙, " make," &c.

159. *Is any reason assigned by Euclid that AC or AB shall meet the* ⊙*ce in C?*

The reason which ought to have been given is that

Since AC is drawn from A between the tangent and ⊙*ce, it must cut the* ⊙*ce* (Prop. XVI., Book III.) *let it cut the* ⊙*ce in C, &c.*

PROP. III.

160. *How does it appear from the text that the tangents will meet in the points M, N, L?*

This ought to have been demonstrated.

If AB be joined, then the ∠s MAB, MBA, being both acute, are together less than two ∟s, and (axiom 12) must meet when produced on the same side of AB with the ∠s. Let them meet in M, &c.

Similarly it may be shown that AL, CL meet, and that BN, CN also meet.

PROP. IV.

161. *Has Euclid shown that BD, CD meet in D?*

In a few words he might have done so. Thus, ∵ the ∠s ABC, ACB are together < two ∟s, much more are ∠s DBC, DCB < two ∟s.

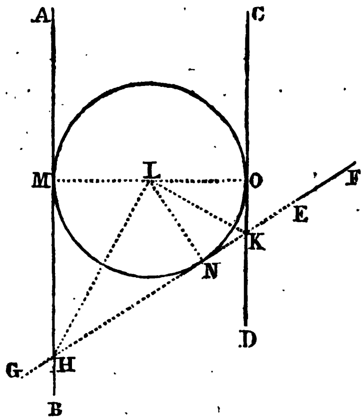
∴ (ax. 12) BD, CD must meet on the same side of BC with the ∠s.

162. *Hence describe a* ⊙ *touching three unlimited* |s *given in position.*

If all three |s are ‖, it is evident that the problem is impossible; for if any (·) be taken in one of them, and from this a | be drawn ⊥ to it, the middle (·) of this ⊥ will be the centre of the ⊙ that shall touch these two |s, but the third | will either cut the ⊙ or lie wholly without it.

If no two are $\parallel$, then if they are produced they must form a $\triangle$. In this $\triangle$ inscribe a $\odot$ and it will touch the three $|$ˢ as required.

If, however, two of them, as AB, CD be $\parallel$, and the third EF not, this third will meet the $\parallel$ lines when produced as in H, K.



Bisect the $\angle$ˢ AHK, CKH by the $|$ˢ HL, KL meeting in L, $\because$ LHK, LKH are together $< AHK + CKH$, and $\therefore$ $<$ two $\llcorner$ˢ.

From L draw $LM \perp AB$ and $LN \perp GF$.

Then $\because$ in the $\triangle$ˢ HML, HNL
the $\angle$ˢ LMH, MHL = $\angle$ˢ LNH, NHL each to each,
and the side LH is common,
$\therefore$ (26, Book I.) $LM = LN$.
Similarly it may be shown that
$LO = LN$,
$\therefore$ LM, LN, LO are $=$,

and the $\odot$ drawn from centre L at the distance LM will pass through M, N, O, and touch the $|$ˢ AB, CD, EF in M, N, O, $\because$ AB, CD, EF are severally $\perp$ to the diameters, &c. &c.

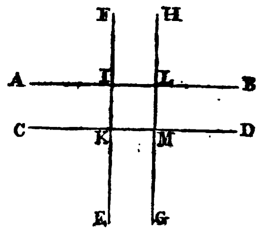
Similarly another tangent $\odot$ may be described on the other side of GF.

Prop. V.

163. Simson has inserted "DF, EF produced meet one another: for if they do not meet, they are $\parallel$; wherefore AB, AC which are at $\llcorner$ˢ to them, are $\parallel$, which is absurd:" has it been shewn in the Elements *that "if two* $|$ˢ *are* $\parallel$, *the* $\perp$ˢ *to them are also* $\parallel$?"

This has not been done.

The Theorem may, however, be thus established.



Let FE, HG be $\perp$ to the $\|^s$ AB, CD. FE is $\|$ to HG.

For let $\angle$ FIA = ∟ and HLI be also a ∟, $\therefore$ exterior $\angle$ = interior $\angle$.

$\therefore$ FE is $\|$ HG.

It is indeed sufficient for the parallelism of two $|^s$ *that they are* $\perp$ *to the same* $|$.

164. *Prove directly that* DF, EF *will meet when produced.*

Suppose (see Euclid's fig.) DE joined.
Then $\because$ $\angle^s$ ADF, AEF = two ∟^{s},
the $\angle^s$ FDE, FED are < two ∟^{s},

$\therefore$ (12 ax.) DF, EF will meet on the same side of DE as that of the $\angle^s$.

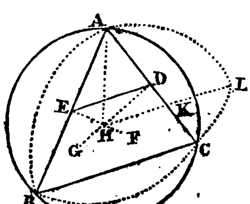
Q. E. D.

165. *Describe a* ⊙ *passing through any three given* (·)^s *which are not all in the same* $|$.

Join the (·)^s, forming thereby a △, and about the △ describe a ⊙, which will be the ⊙ required.

Q. E. F.

166. *No more than one* ⊙ *can be described through the same three given* (·)^s *which are not all in the same* $|$.



Let A, B, C be the three given (·)^s, and if possible, let two ⊙^{s} pass through each of the points A, B, C.

Join AB, AC, and bisect them E and D, and from E and D draw the $\perp^s$ EF, DG, which shall meet in H, $\because$ the $\angle^s$ FED, GDE are together < than the two ∟^{s} FEA, FDA (ax. 12).

But $\because$ EF is at $\llcorner^s$ to the | AC, the centres of both $\odot^s$ are in DG (Prop. I., Cor. Book III.) For the same reason both centres are in EF. They are both $\therefore$ in H, the only common $(\cdot)$ of EF and DG.

Again, $\because$ H is the centre of both $\odot^s$ and A a $(\cdot)$ common to both $\odot^{ces}$, the $\odot^s$ shall coincide and make but one; for if not, let one fall without the other as at L. Join HL, cutting the interior $\odot$ in K, and join HA.

$$\text{Then } \because H \text{ is the centre of } \odot\, AKB,$$
$$\therefore HK = HA.$$
$$\text{and } \because \text{ it is also the centre of the } \odot\, ALB,$$
$$\therefore HL = HA.$$
$$\therefore HL = HK, \text{ the } > = \text{ the } <,$$
$$\text{which is absurd.}$$

$\therefore$ the $\odot^s$ AKB, ALB coincide, and not more than one $\odot$ can pass through the points A, B, C.

Propp. VI and VII.

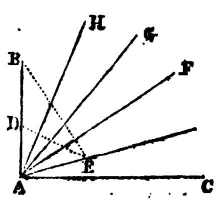
166. From the fig. to these Propositions it is easily seen that *the $\square$ circumscribed about a $\odot$ is double the $\square$ inscribed.*

Prop. IX.

167. It ought to have been shewn that AC, DB do intersect in E.

Prop. X.

168. *By means of this Proposition, quinquesect a $\llcorner$.*



Let BAC be the $\llcorner$.

In AB take any point B, and (Prop. XI., Book II.), divide AB in D, so that

$$AB \times AD = BD^2.$$

With B as centre, and distance BA describe a $\odot$, and in it place $AE = BD$.

$\because$ (Prop. X., Book IV.),

$$\angle\ BAE \text{ and } BEA \text{ each} = 2\angle B,$$

$$\therefore \angle\ BAE + \angle\ BEA + \angle\ B = 5B = \text{two } \llcorner^s.$$

$$\therefore \angle\ B = \tfrac{2}{5} \llcorner.$$

$$\text{But } \angle\ BAE = 2 \angle\ B,$$

$$\therefore \angle\ BAE = \tfrac{4}{5} \llcorner.$$

$$\text{Bisect } \angle\ BAE \text{ by } |\ AG,$$

$$\therefore \angle\ BAG = \angle\ GAE = \tfrac{2}{5} \llcorner.$$

Again, bisect the $\angle^s$ BAG, GAE by AH, AF, and

$$\angle\ BAH = \angle\ HAG = \angle\ GAF = \angle\ FAE = \tfrac{1}{5} \llcorner.$$

$$\text{Hence, } \angle\ BAE = \tfrac{4}{5} \llcorner,$$

$$\text{and } \angle\ EAC = \tfrac{1}{5} \llcorner.$$

$\therefore$ &c.

169. Hence, *a* $\llcorner$ *may be divided into* 10, 20, 40, 80, *&c.* = *parts; viz. by first dividing it into* 5 = *parts; then bisecting these; then each of the* 10 = *parts, &c.*

Also, since a $\llcorner$ may be trisected by means of Propp. I., Book I., and XXXII, Book I., it will easily appear that a $\llcorner$ may be divided into 15, 30, 60, 120, &c. = parts.

Hence, then, by means of bisections, trisections, and quinquesections of a $\llcorner$, it follows that it may be divided into these numbers of = parts, viz.

2, 4, 8, 16, 32, &c.
3, 6, 12, 24, 48, &c.
5, 10, 20, 40, 80, &c.
15, 30, 60, 120, 240, &c.

Prop. XII.

170. *It is not shewn in Simson that BK and CK will meet; how is the omission to be supplied?*

Suppose BC joined. Then, $\because$

$\angle^s$ KBC, KCB are together < KBF, KCF, or < two $\llcorner^s$,

$\therefore$ (ax. 12) BK, CK will meet on the same side of BC as are the $\angle^s$, &c.

Prop. XV.

171. By this and the preceding Propositions it is evident that an △, or any ⊿, a □, a pentagon, a hexagon, an octagon, a decagon, a dodecagon, a quindecagon, a polygon of 16 = sides, a polygon of 20 = sides, &c. and generally a polygon whose number of = sides is denoted by

$$3,\ 2^n,\ \text{or } 2^n \times 3,\ \text{or } 2^n \times 5,\ \text{or } 2^n \times 15,$$

in which n is 2, or > 2, may be inscribed in a ⊙.

But no method has been shewn by Euclid of inscribing regular polygons, whose number of sides is 7, 9, 11, 13, 14, 17, &c.

Gauss, however, has shewn in his *Disquisitiones Arithmeticæ*, that any polygon whose number of sides is of the form $2^n + 1$, provided also that this number is prime, may be inscribed *geometrically* in a ⊙.

His method does not ∴ apply to 9, 33, &c. although it does to 3, 5, 17, &c. sided polygons.

BOOK V.

172. *What is the subject of this Book?*

It contains the doctrine of the Ratios and Proportions of Quantities.

173. *How are Quantities compared as to their relative magnitude?*

In two ways. We frequently institute the comparison by ascertaining the excess of one quantity above another. Thus, if by one road to London from Cambridge the distance is 51 miles, and by another 56 miles, the excess 5 shows the *difference* between these routes, and we thus form a certain comparison.

Another and more perspicuous comparison between two quantities is effected by considering how often the greater contains the less. Thus if one distance be twice as great as another, or three times as great, &c. we have a much clearer idea of their *relative* magnitudes than if merely the excess of the greater were ascertained.

It is this latter species of comparison which Euclid has, therefore, adopted.

Definition I.

174. *What is the distinction between the ordinary acceptation of the word "Part," and that here adopted by Euclid?*

In common language, a part of a magnitude means any portion of it whatever; whereas, Euclid limits the meaning to that portion of the magnitude which is contained in it a number of times exactly.

Thus of the magnitude 18, 7 is a part in common language; 2, 3, 6 are parts of it according to Geometry.

DEFINITION II.

175. *When are Quantities "commensurable?"*

When a third can be found which divides each of them without a remainder.

When are quantities incommensurable?

When there is no quantity which divides each of them exactly.

Thus 6, 24, 9 are commensurable, their common measure or divisor being 3.

Also $\sqrt{6}$, $\sqrt{24}$, $\sqrt{9}$ are commensurable, $\because$ $\sqrt{3}$ is their common measure.

But $\sqrt{6}$ and $\sqrt{11}$ are incommensurable, as are also, 2, $\sqrt{3}$, $\sqrt{5}$, $\sqrt{7}$.

DEFINITION III.

176. According to Dr. Wood (see Alg. p. 91)

"*Ratio* is the relation which one quantity bears to another in respect of magnitude, the comparison being made by considering what multiple, part or parts, one is of the other;" in what does this differ from Euclid's definition.

This definition is much the superior; first, because it implies that *quantities* are compared by finding their *magnitudes,* instead of comparing *magnitudes,* by finding their *quantities;* secondly, because it directs that the comparison be made, not by finding the excess of one above another (see 173) but by "*considering what multiple, part or parts, one is of the other.*"

It is true, Euclid generally means "*quantity*" by the word "*magnitude;*" but as *quantity* in ordinary language expresses indefinitude, or an unmeasured heap, mass, or collection of any thing, and magnitude or number indicates a measured mass,

these terms, as used by Dr. Wood, are more perspicuous. Besides, Euclid's definition not describing in what way the comparison is to be made, and there being two ways, is, in this respect, exceedingly vague and nugatory.

Barrow and others have said much upon this definition to show, that although it is not such in reality, yet it forms a sort of illustrative preliminary to the accurate definitions that succeed it. Dr. Wood, however, has done better by so amending it as to render it a real definition of ratio.

177. *What ought to be the meaning of the words "of the same kind," as used in this definition?*

That (still to use Euclid's sense of *magnitude and quantity)* the magnitudes compared be of the same order of quantity, viz. either both *finite*, both *infinite*, or both *infinitesimal* quantities.

178. *What "quantities," therefore, are "of the same kind?"*

All Finite Quantities; all infinites of the same order; or all infinitesimals of the same order; and no other.

179. *What "magnitudes" are "of the same kind?"*

There are strictly not any, except abstract or ideal magnitudes. Numbers are all "of the same kind," according to the sense of this definition, although they themselves consist of various species; lines are "of the same kind;" superficies are all "of the same kind;" and solids are magnitudes "of the same kind." But magnitudes consisting of actual substance are not necessarily "of the same kind." Indeed, from the endless diversity of nature we can hardly imagine any two substances exactly "of the same kind," nor even that any substance shall, at two consecutive instants of time, be exactly the same thing.

Hence abstract magnitudes are the only homogeneous ones. But abstract magnitudes are the same thing as quantities.

On this subject have been committed great mistakes. For instance, some have asserted that "the diameter of a circle has no exact ratio to the circumference, because they are lines of different kinds;" whereas, since the diameter may be multiplied so as to exceed the circumference (see Def. IV.), or since both are finite, they are "*of the same kind*," and have a ratio to one another.

Hence finite quantities may have a ratio to one another, be the nature of them any whatever; whether they be integers, fractions, surds, exponentials, logarithms, or any other.

The above errors of commentators appear to have arisen from confounding the definition of ratio with that of proportion.

180. *What are the Names of the first and second terms of a Ratio?*

The first is called the *Antecedent*, and the second the *Consequent*.

181. *How is a Ratio* (viz. *of a to b*) *expressed Symbolically?*

Either by $\frac{a}{b}$

or by $a : b$,

in which a is the antecedent, and b the consequent.

Definition V.

182. *What is the "Multiple" of a Magnitude? and what are "Equimultiples?"*

It is the double, triple, quadruple, quintuple, &c. of a magnitude. Thus, of the magnitude A, the quintuple, septuple, decuple are, $5A$, $7A$, $10A$.

Equimultiples of the magnitudes A, B are $5A$, $5B$; $10A$, $10B$; or mA, mB; &c.

183. This definition will be better understood if expressed symbolically; thus,

Of the four magnitudes A, B, C, D, the first A is said to have the same ratio to the second B, which the third C has to the fourth D, when

$$m.A,\ n.B,\ m.C,\ n.D$$

being taken, in which m and n are any numbers whatsoever,

$$\text{if } mA \text{ be } < nB, \text{ then } mC \text{ is } < nD;$$
$$\text{if } mA = nB, \text{ then } mC \text{ is } = nD;$$
$$\text{or, if } mA \text{ be } > nB, \text{ then } mC \text{ is } > nD.$$

Observe;

mA, mC are equimultiples of the first and third, or of A and C,

and

nB, nD are equimultiples of the second and fourth, or of B and D.

184. *What is the common Arithmetical Definition of Proportion?*

Four quantities are said to be proportionals when the quotient arising from the division of the first by the second is exactly the same as the quotient produced by dividing the third by the fourth; whether these quotients be integers, or decimal fractions carried to any number of digits whatever.

For example,

$$\because \tfrac{12}{3} = 4,$$
$$\text{and } \tfrac{108}{27} = 4;$$

$\therefore$ 12, 3, 108, and 27 are proportionals,

$$\text{or } 12 : 3 :: 108 : 27;$$

or as it is otherwise written,

$$12 : 3 = 108 : 27;$$

or again,

$$\tfrac{12}{3} = \tfrac{108}{27}.$$

Example 2.

$$\because \tfrac{3}{8} = \cdot 375,$$
$$\text{and } \tfrac{15}{40} = \cdot 375,$$
$$\therefore 3 : 8 :: 15 : 40.$$

Example 3.

$$\because \tfrac{19}{7} = 2\cdot 7142857142857, \text{ \&c.},$$
$$\tfrac{57}{21} = 2\cdot 7142857142857, \text{ \&c.},$$
$$\therefore 19 : 7 :: 57 : 21.$$

Example 4.

$$\because \frac{8}{\sqrt{128}} = \frac{8}{2\sqrt{32}} = \frac{8}{8\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2},$$
$$\text{and } \frac{16}{\sqrt{512}} = \frac{16}{2\sqrt{128}} = \frac{16}{16\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2};$$

$\therefore$ they must produce the same decimal quotients when $\sqrt{2}$ is extracted, and the root afterwards divided by 2; consequently by the definition,

$$8 : \sqrt{128} :: 16 : \sqrt{512}.$$

Example 5.

$$\because \tfrac{8}{7} = 1\cdot 14285714285, \text{ \&c.},$$
$$\text{and } \tfrac{15}{14} = 1\cdot 07, \text{ \&c.},$$

or produce different quotients;

$\therefore$ 8, 7, 15, 14,
are *not* proportionals.

185. *Why does not this Arithmetical Definition apply to Geometry?*

Because in Geometry the operation of division is not performed according to any particular notation, such as the decimal notation of arithmetic. Lines, &c. are bisected, trisected, &c., but they are conceived to consist of parts whose number is taken in the abstract, without any reference to the system of units, tens, hundreds, &c.

By Definitions V. and VI., four quantities A, B, C, D are proportionals when the two first have the *same* ratio as the two last; that is, when the ratio $A : B$ is the same as that of $C : D$; that is, when

$$A : B = C : D.$$

Hence, analogy is the *equality* of ratios; so that "similitude of ratios" means nothing else than *equality of ratios*. This latter term being much more intelligible, because less vague, ought to have been used, unless indeed the definition had been altogether omitted.

Definition IX.

188. *How is it that three Terms may constitute a Proportion?*

This may be, provided the *consequent* of the first ratio be the same as the *antecedent* of the second ratio.

Thus, if $A : B$, and $B : C$ be equal ratios, then

$$A : B :: B : C.$$

189. *What is a "Mean Proportional?"*

If three quantities be proportionals, the second is termed the mean proportional. Thus, if

$$A : B :: B : C,$$

then B is a mean proportional to A and C.

190. *How many Terms are necessary to exhibit a Ratio?*

Any one term, as A, exhibits the ratio

$$A : 1,$$

$\because$ A is supposed to consist of a certain number of units or parts of unit. All magnitudes are supposed comparable with unit either exactly or approximately to any required degree of accuracy. Consequently, A denotes the "mutual relation of A to 1, in respect of quantity," and $\therefore$ the ratio $A : 1$.

Definitions X., XI., and A.

191. *What is to be understood by "Compound Ratio?"*

Def. A, as given by Simson, means that if

$$A,\ B,\ C,\ D,\ E,\ F,\ G,\ H,\ K,\ L$$

be continual proportionals; that is, if the ratios
$A : B$, $B : C$, $C : D$, $D : E$, $E : F$, $G : H$, $H : K$, $K : L$
be all equal, then the ratio

$$A : L$$

is that which is compounded of all the above ratios.

Hence,

192. A "*Duplicate Ratio*"
is that which is compounded of *two* equal ratios, as of $A : B$, $B : C$; viz. $A : C$.

A "*Triplicate Ratio*"
is that which is compounded of *three* equal ratios, as of $A : B$, $B : C$, $C : D$; viz. $A : D$.

A "*Quadruplicate Ratio*"
is that which is compounded of *four* equal ratios, as of $A : B$, $B : C$, $C : D$, $D : E$; viz. $A : E$.

A "*Quintuplicate Ratio*"
is that which is compounded of *five* equal ratios, and so on.

Hence it also follows, that if any number of magnitudes be continual proportionals,

The ratio compounded of a *duplicate* ratio and of a *triplicate* ratio will be a *quintuplicate* ratio; that

The ratio compounded of a *duplicate* ratio, a *triplicate* ratio, and a *quintuplicate* ratio, is a *decuplicate* ratio, and so on.

Also it appears that every compound ratio of any number of continual proportionals is the ratio of the first to the last of the proportionals.

192. *How are Ratios compounded Arithmetically?*

By multiplying the antecedents together, and the consequents together; the ratio compounded of the several simple

ratios will be that whose antecedent is the product of the antecedents, and consequent, the product of the consequents.

Example 1.

If A, B, C be proportionals, then

$$A : B :: B : C;$$

or (see 181),

$$\left.\begin{aligned} \frac{A}{B} &= \frac{B}{C} \\ \text{But } \frac{A}{B} &= \frac{A}{B} \end{aligned}\right\}$$

and multiplying these equals, we get

$$\frac{A^2}{B^2} = \frac{B}{C} \times \frac{A}{B} = \frac{A}{C}.$$

$$\therefore A : C :: A^2 : B^2;$$

or A is to C in the duplicate ratio of A to B.

Example 2.

If A, B, C, D be continual proportionals, then

$$A : B = B : C = C : D.$$

or,

$$\left.\begin{aligned} \frac{A}{B} &= \frac{B}{C} \\ \frac{A}{B} &= \frac{C}{D} \\ \text{and } \frac{A}{B} &= \frac{A}{B} \end{aligned}\right\}$$

$\therefore$ multiplying these three sets of equals together, we get

$$\frac{A^3}{B^3} = \frac{B}{C} \times \frac{C}{D} \times \frac{A}{B} = \frac{A}{D}.$$

$$\therefore A : D :: A^3 : B^3,$$

or A is to D in the triplicate ratio of A to B.

Example 3.

If A, B, C, D, E be continual proportionals, then

$$\frac{A}{B}, \frac{B}{C}, \frac{C}{D}, \frac{D}{E} \text{ are all equal,}$$

$$\text{and } \frac{A}{B} \times \frac{B}{C} \times \frac{C}{D} \times \frac{D}{E} = \frac{A}{B} \times \frac{A}{B} \times \frac{A}{B} \times \frac{A}{B};$$

$$\therefore \frac{A}{E} = \frac{A^4}{B^4},$$

or A is to E in the quadruplicate ratio of A to B, and so on.

DEFF. XIV., XV., XVI., XVII., XVIII., XIX., XX., XXI.

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193. The various rules laid down by these definitions for the changes that may take place in the terms of a proportion may be exhibited in one view by the following

PROPORTION TABLE.

If							
	A	:	B	::	C	:	D
Permutando	A	:	C	::	B	:	D
Invertendo	B	:	A	::	D	:	C
Componendo	$A+B$	:	B	::	$C+D$	:	D
Dividendo	$A-B$	:	B	::	$C-D$	:	D
Convertendo	A	:	$A+B$	::	C	:	$C+D$
or	A	:	$A-B$	::	C	:	$C-D$
Also if	A	:	B	::	D	:	E
and	B	:	C	::	E	:	F
Ex æquo ordinatè	A	:	C	::	D	:	F
and if	A	:	B	::	E	:	F
and	B	:	C	::	D	:	E
Ex æquo perturbatè	A	:	C	::	D	:	F

ADDITIONAL FORMS

USED IN

ARITHMETIC AND ALGEBRA.

$$A+B : A-B :: C+D : C-D$$

$$m.A : m.B :: n.C : n.D$$

Prop. IV., Book V., $m.A : n.B :: m.C : n.D$

$$\frac{A}{m} : \frac{B}{m} :: \frac{C}{n} : \frac{D}{n}$$

$$\frac{A}{m} : \frac{B}{n} :: \frac{C}{m} : \frac{D}{n}$$

$$\frac{p}{q}.A : \frac{p}{q}.B :: \frac{m}{n}.C : \frac{m}{n}.D$$

$$\frac{p}{q}.A : \frac{m}{n}.B :: \frac{p}{q}.C : \frac{m}{n}.D$$

$$A^m : B^m :: C^m : D^m$$

$$A^{\frac{1}{m}} : B^{\frac{1}{m}} :: C^{\frac{1}{m}} : D^{\frac{1}{m}}$$

$$A^{\frac{p}{q}} : B^{\frac{p}{q}} :: C^{\frac{p}{q}} : D^{\frac{p}{q}}$$

If
$$\left.\begin{array}{l} A : B :: C : D \\ P : Q :: R : S \\ W : X :: Y : Z \end{array}\right\}$$

then

$$A.P.W : B.Q.X :: C.R.Y : D.S.Z$$

&c. &c.

194. In Newton's "Principia," and other works on Natural Philosophy, treated Geometrically, the following different kinds of ratios are to be found, as well as those specified in Euclid, viz.

RATIO OF EQUALITY,
CONSTANT OR GIVEN RATIO,

DIRECT RATIO,
INVERSE OR RECIPROCAL RATIO,
SUBDUPLICATE RATIO,
SUBTRIPLICATE RATIO,
SESQUIPLICATE RATIO.

Now Euclid being merely a preliminary step to Philosophy, it may not be amiss to explain the meaning of these several technical terms.

195. *What is meant by "Ratio of Equality?"*

A ratio of equality is that in which the antecedent and consequent are equal. Thus $A : A$ is a ratio of equality.

196. *What is a "Constant or Given Ratio?"*

Two magnitudes, mA and mB, which for all values of m constitute the same ratio, are said to be in a given ratio, viz. that of $A : B$.

197. *What is the distinction between Direct and Inverse, or Reciprocal Quantities?*

Taking the words "magnitude" and "Quantity," in the sense of Euclid (see Def. III., Book V.), a magnitude (A) may be any thing whatever having extension; but if this thing or *magnitude* be considered as to *quantity*, it must be compared with the unit of its species; that is, the ratio $A : 1$ or $\frac{A}{1}$ denotes the *quantity* of the magnitude A.

Hence the magnitude A being compared with its unit, gives the quantity $\frac{A}{1}$, or, as it is written for the sake of brevity, A. But if we make the unit the antecedent, and A the consequent, or compare 1 with A, the resulting ratio is $\frac{1}{A}$; and these quantities being formed in an inverse or reciprocal order, the one is called the inverse or reciprocal of the other.

$$\text{Hence } \frac{1}{A} \text{ is the reciprocal of } A,$$

$$\text{and } A \text{ is the reciprocal of } \frac{1}{A}.$$

From the above reasoning it also appears that *all ratios are quantities;* and that magnitudes are nothing but certain extensions.

Example.

A certain *unmeasured* area (A) is proposed for consideration. This is a *Magnitude.*

Let it be measured, and suppose that it contain 1000 square inches. Then the unit of surface being supposed 1 square inch, we have

$$\frac{A}{1} = 1000;$$

$$\text{and } \frac{A}{1} \text{ is a quantity, viz. } 1000.$$

198. *What is a Direct Ratio?*

That in which the antecedent and consequent are either both direct quantities, or both inverse quantities.

$$A : B \text{ or } \frac{A}{B},$$

also,

$$\frac{1}{A} : \frac{1}{B}, \text{ or } \frac{\left(\frac{1}{A}\right)}{\left(\frac{1}{B}\right)},$$

are direct ratios.

199. *What is an Inverse, or Reciprocal Ratio?*

That in which the antecedent and consequent are neither both direct quantities, nor both inverse quantities.

$$A : \frac{1}{B} \text{ or } \frac{A}{\left(\frac{1}{B}\right)},$$

$$\text{Also, } \frac{1}{A} : B \text{ or } \frac{\left(\frac{1}{A}\right)}{B},$$

are inverse or reciprocal ratios.

200. *What is a Subduplicate Ratio?*

If three magnitudes A, B, C, be proportionals, then A is said to have to C the *duplicate* ratio of that which A has to B (see Def. X.); and conversely,

A is said to have to B the subduplicate ratio of that which A has to C.

Algebraically exhibited.

If A, B, C be proportionals, then (192)

$$A : C :: A^2 : B^2 \text{ or } \frac{A}{C} = \frac{A^2}{B^2};$$

and extracting the roots of these equals,

$$\frac{A}{B} = \frac{A^{\frac{1}{2}}}{C^{\frac{1}{2}}};$$

$$\text{or } A : B :: A^{\frac{1}{2}} : C^{\frac{1}{2}}.$$

Hence then

A subduplicate ratio is such, that, of three proportionals, the first is said to have to the second the subduplicate *ratio of that which the first has to the third.*

201. *What is a Subtriplicate Ratio?*

As before, the reasons may be manifested why it should be defined,

Of four proportionals the first is said to have to the second the subtriplicate *ratio of that which the first has to the fourth.*

Thus, in algebraic notation, if A, B, C, D be proportionals; then

$$A : B :: A^{\frac{1}{3}} : C^{\frac{1}{3}}.$$

Similarly may subquadruplicate ratio, subquintuplicate ratio, &c. be defined.

202. *What is a Sesquiplicate Ratio?*

It is the ratio compounded of the simple ratio and of the subduplicate ratio (see Def. XI. and *A*, Book V.)

In algebraic notation it is thus formed;

$$\text{Let } \frac{A}{B} \text{ be the simple ratio.}$$

$$\text{Then } \frac{\sqrt{A}}{\sqrt{B}} \text{ is the subduplicate ratio;}$$

$\therefore$ the sesquiplicate ratio is

$$\frac{A}{B} \times \frac{\sqrt{A}}{\sqrt{B}} = \frac{A^{\frac{3}{2}}}{B^{\frac{3}{2}}}.$$

Prop. I.

203. *In the demonstration of this Proposition Euclid says, "Divide AB into magnitudes equal to E:" how is this to be done?*

By first cutting off (Prop. III., Book I.) $AG = E$, then $GB = E$, and so on, until the multiple is exhausted.

204. Although it is not permitted students to use algebra in Book V. in their demonstrations, yet it may aid them considerably in comprehending the enunciations, and the several steps of the geometrical proof. For this reason we shall give the several propositions by algebra; or, at any rate, symbolically, preserving the logic of Euclid unaltered.

Prop. I. *Symbolically.*

If A, B, C, &c. be equimultiples of a, b, c, &c., then $A + B + C +$ &c. is the same multiple of $a + b + c +$ &c.

Let A, divided into parts $= a$, be supposed to contain m of them. Then

$$A = ma.$$

And since B, C, &c. are the same multiples of their parts b, c, &c. that A is of a,

$$\therefore B = mb,$$
$$C = mc,$$
$$\&c.$$

$$\begin{aligned}\therefore A + B + C + \&c. &= ma + mb + mc + \&c.\\ &= a + a + \&c.\text{ to } m \text{ terms},\\ &+ b + b + \&c.\text{ to } m \text{ terms},\\ &+ c + c + \&c.\text{ to } m \text{ terms}\\ &= (a + b + c + \&c.) + (a + b + c\\ &\qquad + \&c.)\text{ to } m \text{ terms}\\ &= m\,(a + b + c + \&c.)\end{aligned}$$

Q. E. D.

This proof, although conducted by the aid of algebraic symbols, is the same in reasoning as Euclid's.

Prop. II. *Symbolically.*

205. *If A, a, B, b, A', B' be six magnitudes, and A, B equimultiples of a and b, and A', B' also equimultiples of a and b, then $A + A'$, $B + B'$ shall be equimultiples of a and b.*

For if A and B contain a and b respectively m times, and A', B' contain a and b respectively n times,

then
$$A = m \times a,\quad B = m \times b,$$
$$A' = n \times a,\quad B' = n \times b;$$

and $A + A' = m \times a + n \times a = (m + n)\,a$,
and $B + B' = m \times b + n \times b = (m + n)\,b$;

that is

$A + A'$ and $B + B'$ are equimultiples of a and b.

Prop. II. Cor.

206. If $A = m \times a$, $A' = n \times a$, $A'' = p \times a$, $A''' = q \times a$, &c., and $B = m \times b$, $B' = n \times b$, $B'' = p \times b$, $B''' = q \times b$, &c.,

then

$$A+A'+A''+\&c. = ma+na+pa+\&c. = (m+n+p+\&c.)\,a$$
$$B+B'+B''+\&c. = mb+nb+pb+\&c. = (m+n+p+\&c.)\,b.$$

$\therefore$ $A + A' + A'' + \&c.$, $B + B' + B'' + \&c.$ are equimultiples of a and b.

Q. E. D.

Prop. III. *Algebraically.*

207. *If of four magnitudes A, a, B, b the first A and third B are equimultiples of a and b, then if equimultiples of A and B be taken, these shall also be equimultiples of a and b.*

Let A contain a, m times, and B contain b, m times; then

$$A = m \times a,$$
$$\text{and } B = m \times b,$$

and if we take the equimultiples of A and B such that they shall contain A and B, p times, they are

$$p \times A,\ p \times B.$$

But A, B contain a and b, m times.

$\therefore$ pA and pB contain a and b, $p \times m$ times,

$$\text{or } pA = pma$$
$$pB = pma,$$

and pA, pB are equimultiples of a and b.

Q. E. D.

Prop. IV.

208. *If $A : a :: B : b$; then $mA : mB :: na : nb$.*

For take any equimultiples whatever of mA, mB, as $M.mA$, $M.mB$, and any whatever of na and nb, as $N.na$, $N.nb$.

Then MmA, MmB are equimultiples of A and B, and Nna, Nnb are equimultiples of a and b; and $\because$

$$A : a :: B : b,$$

$\therefore$ if MmA be $> Mmb$, Nna is $> Nnb$, and if equal, equal; and if less, less (Def. V., Book V.) But MmA, MmB are any equimultiples whatever of mA and mB, and Nna, Nnb are any whatever of na and nb.

$$\therefore mA : mB :: na : nb.$$

Q. E. D.

Algebraically.

$$\text{If } A : a :: B : b.$$

$$\frac{A}{a} = \frac{B}{b}.$$

$$\therefore \frac{A}{B} = \frac{a}{b}.$$

$$\therefore \frac{mA}{mB} = \frac{na}{nb}.$$

$$\therefore mA : mB :: na : nb.$$

Q. E. D.

The Corollary to this Proposition is contained in the Proposition itself, being that case in which $n = 1$.

Prop. V.

209. *If A is the same multiple of a that B is of b, then $A - B$ is the same multiple of $a - b$.*

For let

$$A = ma, \ B = mb,$$

Then

$$A - B = ma - mb$$
$$= m(a - b).$$

Q. E. D.

Prop. VI.

210. *If A, B be equimultiples of a and b, then $A - ma$, $B - mb$ are equimultiples of a, b, or are $=$ to them.*

For let $A = na$, $B = nb$; then

$$A - ma = na - ma = (n - m)\,a,$$
$$\text{and } B - mb = nb - mb = (n - m)\,b.$$

Q. E. D.

When $n = 2$, and $m = 1$,

then

$$\left.\begin{array}{r} A - a = a \\ \text{and } B - b = b \end{array}\right\}\text{which is the particular case of Euclid.}$$

211. *Ought there not to have been another case of this Proposition?*

When the equimultiples taken of a and b are equal to A and B respectively, or when $m = n$, then

$$A - ma = (m - m)\,a = 0,$$
$$\text{and } B - mb = (m - m)\,b = 0.$$

Q. E D.

Prop. A.

212. *If $A : a :: B : b$, then, according as*
A is $>$, $=$, or $<$ a, B is $>$, $=$, or $<$ b.

For if any equimultiples of each of them be taken, as

$$mA,\ ma,\ mB,\ mb,$$

then (Def. V., Book V.), according as

mA is $>$, $=$, or $<$ ma, mB is $>$, $=$, or $<$ mb.

But if A be $>$, $=$, or $<$ a, then mA is $>$, $=$, or $<$ ma.

$\therefore$ mB is $>$, $=$, or $<$ mb.

$\therefore$ B is $>$, $=$, or $<$ b.

$\therefore$ according as

A is $>$, $=$, or $<$ a, B is $>$, $=$, or $<$ b.

Q. E. D.

Prop. B.

213. *If $A : a :: B : b$; then $a : A :: b : B$ (Invertendo.)*

$$\text{For } \frac{A}{a} = \frac{B}{b};\ \therefore \frac{a}{A} = \frac{b}{B}.$$

$$\therefore a : A :: b : B.$$

Q. E. D.

Prop. C.

214. *If* $A = ma$, $B = mb$, *or if* $A = \frac{a}{m}$, *and* $B = \frac{b}{m}$,

then $A : a :: B : b$.

Q. E. D.

For

$$\frac{A}{a} = m, \text{ and } \frac{B}{b} = m.$$

$$\therefore \frac{A}{a} = \frac{B}{b}.$$

$$\therefore A : a :: B : b.$$

Q. E. D.

Prop. D.

215. *If* $A : a :: B : b$, *and* $A = ma$ *or* $\frac{a}{m}$, *then*

also, $B = mb$, *or* $= \frac{b}{m}$.

For

$$\frac{B}{b} = \frac{A}{a},$$

$$\text{and } \frac{A}{a} = m \text{ or } \frac{1}{m}.$$

$$\therefore \frac{B}{b} = m \text{ or } \frac{1}{m}.$$

$$\therefore B = mb \text{ or } \frac{b}{m}.$$

Q. E. D.

Prop. VIII.

216. *If* A *be* $> A'$, *then*

$A : a$ *is* $> A' : a$,

and $a : A'$ *is* $> a : A$.

$$\text{For } \frac{A}{a} \text{ is } > \frac{A'}{a}.$$

$\because$ the quotient arising from the division (see Art. on Def. V., Book V.) of A by a is $>$ the quotient of $A \div a$.

$$\text{Also } \frac{a}{A'} \text{ gives a greater quotient than}$$

$$\frac{a}{A}, \because \text{ the dividend being the same,}$$

that which has the less divisor must produce a greater quotient.

Q. E. D.

PROP. IX.

217. *If $A : a$ is the same as $A' : a$, then $A = A'$; and if $a : A$ is the same as $a : A'$, then $A = A'$.*

$$\text{For } \because \frac{A}{a} = \frac{A'}{a},$$

$$\therefore A = A.$$

$$\text{Also } \because \frac{a}{A} = \frac{a}{A'},$$

$$\therefore \frac{1}{A} = \frac{1}{A'}.$$

$$\therefore A' = A.$$

Q. E. D.

PROP. X.

218. *If $A : a$ be $> A' : a$, then A is $> A'$; and if $a : A$ be $> a : A'$, then A is $< A'$.*

$$\text{For if } \frac{A}{a} \text{ be } > \frac{A'}{a},$$

the quotient $A \div a$ is $>$ quotient $A' \div a$; but the divisor being the same, the dividend A must be $>$ dividend A'.

Again, if

$$\frac{a}{A} \text{ be } > \frac{a}{A'},$$

the quotient $a \div A$ is $>$ quotient $a \div A'$, and the dividend being the same, the divisor A must be $<$ divisor A'.

Q. E. D.

Prop. XI.

219. *If* $A : a :: B : b$, *and* $B : b :: C : c$, *then* $A : a :: C : c$.

$$\text{For } \frac{A}{a} = \frac{B}{b},$$

$$\text{and } \frac{B}{b} = \frac{C}{c}.$$

$$\therefore \frac{A}{a} = \frac{C}{c}.$$

$$\therefore A : a :: C : c.$$

Q. E. D.

Prop. XII.

220. *If* $A : a :: B : b :: C : c ::$ *&c.*,

then

$A : a :: A + B + C +$ &c. $: a + b + c +$ &c.

$$\text{For } \frac{A}{a} = \frac{B}{b}.$$

$$\therefore \frac{A}{B} = \frac{a}{b}.$$

$$\therefore \frac{A}{B} + 1 = \frac{a}{b} + 1.$$

$$\therefore \frac{A + B}{B} = \frac{a + b}{b}.$$

$$\therefore \frac{A + B}{a + b} = \frac{B}{b} = \frac{C}{c} = \frac{A}{a}, \text{ \&c.}$$

Q. E. D.

Again,

$$\because \frac{A + B}{a + b} = \frac{C}{c},$$

$$\therefore \frac{A+B}{C} + 1 = \frac{a+b}{c} + 1.$$

$$\therefore \frac{A+B+C}{C} = \frac{a+b+c}{c}.$$

$$\therefore \frac{A+B+C}{a+b+c} = \frac{C}{c} = \frac{A}{a}.$$

$$\therefore A : a :: A + B + C : a + b + c;$$

and the process being continued, the proposition may be demonstrated for any number of antecedents and consequents.

Q. E. D.

Prop. XIII.

221. *If $a : A :: B : b$, but $B : b > C : c$;*

then

$$A : a \text{ is } > C : c.$$

For $\because B : b$ is $> C : c$,

and $A : a = B : b$,

$$\therefore A : a \text{ is } > C : c.$$

Q. E. D.

Prop. XIV.

222. *If $A : a :: B : b$, then if A be $> B$, a is $> b$; and if $A = B$, then $a = b$; and if A be $< B$, then a is also $< b$.*

For

$$\frac{A}{a} = \frac{B}{b};$$

$$\therefore \frac{A}{B} = \frac{a}{b};$$

$$\therefore A : B :: a : b.$$

But, when magnitudes are proportionals, if the first be $>$

the second, the third is also > the fourth, and if equal, equal, and if less, less (Def. V. Book V.)

$\therefore$ If, &c.

Q. E. D.

PROP. XV.

223. $A : a :: m.A : m.a.$

For

$$\frac{A}{a} = \frac{mA}{ma}.$$

Q. E. D.

PROP. XVI.

224. *If* $A : a :: B : b$; *then* $A : B :: a : b$ *alternando.*

$$\text{For } \because \frac{A}{a} = \frac{B}{b},$$

$$\therefore \frac{A}{B} = \frac{a}{b},$$

&c.

Q. E. D.

PROP. XVII.

225. *If* $A + B : B :: a + b : b$;

then

$$A : B :: a : b.$$

For

$$\frac{A+B}{B} = \frac{a+b}{b};$$

$$\therefore \frac{A+B}{B} - 1 = \frac{a+b}{b} - 1,$$

$$\text{or } \frac{A}{B} + 1 - 1 = \frac{a}{b} + 1 - 1,$$

$$\text{or } \frac{A}{B} = \frac{a}{b}.$$

$$\therefore A : B :: a : b;$$

and similarly for any number of proportionals.

Q. E. D.

Prop. XVIII.

226. *If $A : a :: B : b$, then $A + a : a :: B + b : b$.*

For

$$\frac{A}{a} = \frac{B}{b};$$

$$\therefore \frac{A}{a} + 1 = \frac{B}{b} + 1;$$

$$\therefore \frac{A + a}{a} = \frac{B + b}{b};$$

$$\text{or } A + a : a :: B + b : b.$$

Q. E. D.

Prop. XIX.

227. *If $A + a : B + b :: a : b$; then $A : B :: A + a : B + b$.*

For

$$\frac{A + a}{B + b} = \frac{a}{b};$$

$$\therefore \frac{A + a}{a} = \frac{B + b}{b};$$

$$\text{or } \frac{A}{a} + 1 = \frac{B}{b} + 1;$$

$$\therefore \frac{A}{a} = \frac{B}{b};$$

$$\therefore \frac{A}{B} = \frac{a}{b} = \frac{A + a}{B + b},$$

$$\text{or } A : B :: A + a : B + b.$$

Q. E. D.

PROP. E.

228. *If* $A : a :: B : b$*; then* $A : A \sim a :: B : B \sim b$.

For

$$\frac{A}{a} = \frac{B}{b};$$

$$\therefore \frac{A}{a} \sim 1 = \frac{B}{b} \sim 1;$$

$$\therefore \frac{A \sim a}{a} = \frac{B \sim b}{b},$$

$$\text{or } \frac{A \sim a}{B \sim b} = \frac{a}{b} = \frac{A}{B};$$

$$\therefore \frac{A}{A \sim a} = \frac{B}{B \sim b}.$$

$$\therefore A : A \sim a :: B : B \sim b.$$

Q. E. D.

PROP. XX.

229. *If* A, a, α *be three magnitudes, and* B, b, β *three others, and if* $A : a :: B : b$, *and* $a : \alpha :: b : \beta$*; then, if* A *be* $> \alpha$, B *is also* $> \beta$, *and if equal, equal; and if less, less.*

For

$$\frac{A}{a} = \frac{B}{b},$$

and

$$\frac{a}{\alpha} = \frac{b}{\beta},$$

$$\therefore \frac{A}{a} \times \frac{a}{\alpha} = \frac{B}{b} \times \frac{b}{\beta};$$

$$\therefore \frac{A}{\alpha} = \frac{B}{\beta};$$

$$\text{or } A : \alpha :: B : \beta;$$

$\therefore$ (Def. V., Book V.), if A be $> \alpha$, B is also $> \beta$; if equal, equal; and if less, less. Q. E. D.

Prop. XXI.

230. *If A, a, α; B, b, β be such that $A : a :: b : \beta$, and $a : \alpha :: B : b$; then if A be $> \alpha$, B is also $> \beta$, and if equal, equal; and if less, less.*

For

$$\frac{A}{a} = \frac{b}{\beta},$$

$$\text{and } \frac{a}{\alpha} = \frac{B}{b};$$

$$\therefore \frac{A}{a} \times \frac{a}{\alpha} = \frac{b}{\beta} \times \frac{B}{b};$$

$$\therefore \frac{A}{\alpha} = \frac{B}{\beta};$$

or $A : \alpha :: B : \beta$, $\therefore$ (Def. V., Book V.), if A be $> \alpha$, B is $> \beta$, &c.

Q. E. D.

Prop. XXII.

231. *If A, A', A'', &c. be any Magnitudes, and B, B', B'', &c. be as many others, such that*

$$A : A' :: B : B'$$
$$A' : A'' :: B' : B''$$
$$A'' : A''' :: B'' : B'''$$
&c.

Then ex æquo

$$A : A''''\cdots :: B : B''''\cdots.$$

For

$$\frac{A}{A'} = \frac{B}{B'},$$

$$\frac{A'}{A''} = \frac{B'}{B''},$$

$$\frac{A''}{A'''} = \frac{B''}{B'''},$$

&c.

$$\therefore \frac{A}{A'} \times \frac{A'}{A''} \times \frac{A''}{A'''}, \text{ \&c.} = \frac{B}{B'} \times \frac{B'}{B''} \times \frac{B''}{B'''} \times \text{ \&c.},$$

$$\text{or } \frac{A}{A'''\ldots} = \frac{B}{B'''\ldots};$$

$$\therefore A : A'''\ldots :: B : B'''\ldots$$

Q. E. D.

In Geometrical investigations it is usual, in cases of *Ex æquo*, thus to cancel the intermediate antecedents and consequents. If

$$A : A' :: B : B'$$
$$A' : A'' :: B' : B''$$
$$A'' : A''' :: B'' : B'''$$

&c.

$$\therefore A : A'''\ldots :: B : B'''\ldots$$

Prop. XXIII.

232. *If A, A', A'', &c. be any Magnitudes, and B, B', B'', &c. as many others, such that (taking three of each for example)*

$$A : A' :: B'' : B'''$$
$$A' : A'' :: B' : B''$$
$$A'' : A''' :: B : B',$$

then *ex æquo perturbato*

$$A : A''' :: B : B'''.$$

For

$$\frac{A}{A'} = \frac{B''}{B'''},$$

$$\frac{A'}{A''} = \frac{B'}{B''},$$

$$\frac{A''}{A'''} = \frac{B}{B'}.$$

$$\therefore \frac{A}{A'} \times \frac{A'}{A''} \times \frac{A''}{A'''} = \frac{B''}{B'''} \times \frac{B'}{B''} \times \frac{B}{B'},$$

$$\text{or } \frac{A}{A'''} = \frac{B}{B'''}.$$

$$\therefore A : A''' :: B : B''';$$

and in the same way it may be demonstrated for a greater number of magnitudes.

Q. E. D.

The cancelling is effected as above described, by drawing the pen crosswise through such antecedents and consequents as are identical.

PROP. XXIV.

233. *If* $A : a :: B : b$, *and* $A' : a :: B' : b$, *then*
$A + A' : a :: B + B' : b.$

For

$$\frac{A}{a} = \frac{B}{b},$$

$$\text{and } \frac{A'}{a} = \frac{B'}{b}.$$

$$\therefore \frac{A}{B} = \frac{a}{b} = \frac{A'}{B'}.$$

$$\therefore \frac{A}{A'} = \frac{B}{B'}.$$

$$\therefore \frac{A}{A'} + 1 = \frac{B}{B'} + 1.$$

$$\therefore \frac{A + A'}{A'} = \frac{B + B'}{B'}.$$

$$\therefore \frac{A + A'}{B + B'} = \frac{A'}{B'} = \frac{a}{b}.$$

$$\therefore \frac{A + A'}{a} = \frac{B + B'}{b},$$

or $A + A' : a :: B + B' : b.$

Q. E. D.

Prop. XXIV. Cor. 1.

234. *If* $A : a :: B : b$, *and* $A' : a :: B' : b$; *then*
$$A \sim A' : a :: B \sim B' : b.$$

As before,

$$\frac{A}{A'} \sim 1 = \frac{B}{B'} \sim 1,$$

&c. as is evident.

Prop. XXV.

235. *If* $A : a :: B : b$, *and* A *the greatest of them, then* $A + b$ *is* $> a + B$.

For

$$\frac{A}{a} = \frac{B}{b}.$$

$$\therefore A = \frac{B}{b}.a.$$

$$\therefore A + b = \frac{B}{b}.a + b.$$

$$\therefore A + b - (a + B) = \frac{Ba}{b} + b - a - B$$

$$= B.\left(\frac{a}{b} - 1\right) + b - a$$

$$= B.\frac{a - b}{b} - (a - b)$$

$$= \left(\frac{B}{b} - 1\right)(a - b)$$

$$= \frac{(B - b)(a - b)}{b}.$$

Now if A be $> a$, and also $> B$, then (Def. V., Book V.) B is $> b$, and a (alternando) is $> b$.

$\therefore B - b$, and $a - b$ are both positive.

$\therefore A + b$ is $> a + B$ by $\dfrac{(B - b)(a - b)}{b}$.

Q. E. D.

236. *Is the sum of the Extremes > sum of the Means in all cases?*

If A be $> a$, but $< B$, then a is $< b$,
and $a - b$ is negative, whilst $B - b$ is positive.

$$\therefore A + b \text{ is } < a + B \text{ by } \frac{(B-b)(b-a)}{b}.$$

Prop. F.

237. $$\textit{If } \left.\begin{array}{l} A : B :: a : b \\ B : C :: b : c \\ C : D :: c : d, \end{array}\right\} \textit{ or if } \left.\begin{array}{l} A : B :: c : d \\ B : C :: b : c \\ C : D :: a : b, \end{array}\right\}$$

&c.

$$\textit{then } A : D :: a : d.$$

This is evident *ex æquo* and *ex æquo perturbato*.

Prop. G.

238. $$\textit{If } \left.\begin{array}{l} A : B :: a : b \\ \textit{and } C : D :: c : d, \end{array}\right\} \textit{ and if } \left.\begin{array}{l} A : B :: K : L \\ \textit{and } C : D :: L : M, \end{array}\right\}$$

$$\textit{and if } \left.\begin{array}{l} a : b :: k : l \\ \textit{and } c : d :: l : m, \end{array}\right\}$$

$$\textit{then } K : M :: k : m.$$

$$\text{For } \because \frac{A}{B} = \frac{K}{L}, \text{ and } \frac{C}{D} = \frac{L}{M},$$

$$\therefore \frac{A}{B} \times \frac{C}{D} = \frac{K}{L} \times \frac{L}{M} = \frac{K}{M}.$$

$$\text{Similarly } \frac{a}{b} \times \frac{c}{d} = \frac{k}{m}.$$

$$\text{But } \frac{A}{B} = \frac{a}{b}, \text{ and } \frac{C}{D} = \frac{c}{d}.$$

$$\therefore \frac{A}{B} \times \frac{C}{D} = \frac{a}{b} \times \frac{c}{d}.$$

$$\text{Hence } \frac{K}{M} = \frac{k}{m},$$
$$\text{or } K : M :: k : m.$$

Q. E. D.

239. Propositions H and K are not read at the Universities.

Instead of these, Playfair, in his edition of the "Elements," gives this more useful proposition, viz.

If $A : B :: a : b$; *then* $A + B : A \sim B :: a + b : a \sim b$.

$$\because \frac{A}{B} = \frac{a}{b},$$
$$\therefore \frac{A}{B} + 1 = \frac{a}{b} + 1,$$
$$\text{or } \frac{A + B}{B} = \frac{a + b}{b}.$$

Also,

$$\frac{A}{B} \sim 1 = \frac{a}{b} \sim 1.$$
$$\therefore \frac{A \sim B}{B} = \frac{a \sim b}{b}.$$

Hence

$$\frac{A + B}{a + b} = \frac{B}{b},$$
$$\text{and } \frac{A \sim B}{a \sim b} = \frac{B}{b}.$$
$$\therefore \frac{A + B}{a + b} = \frac{A \sim B}{a \sim b}.$$
$$\therefore \frac{A + B}{A \sim B} = \frac{a + b}{a \sim b},$$
$$\text{or } A + B : A \sim B :: a + b : a \sim b.$$

Q. E. D.

BOOK VI.

240. *What is the subject of Book VI.?*

The comparison of the sides and areas of certain rectilineal figures, as also the investigation of $|^s$ having a required ratio to given $|^s$.

DEFINITIONS.

Definition I.

241. *Is any condition, expressed in this Definition, superfluous when applied to Triangles?*

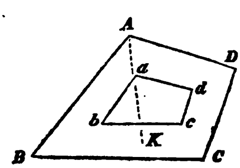
Since (Prop. IV., Book VI.) the sides about the = $\angle^s$ of equiangular $\triangle^s$ are proportionals, it is sufficient for the definition of similar $\triangle^s$ to say

Similar $\triangle^s$ are those which have their $\angle^s$ = each to each; or the best definition that can be given is,

Similar $\triangle^s$ are those in which two $\angle^s$ of the one = two $\angle^s$ of the other, each to each.

That the third $\angle^s$ are =, is evident from Prop. XXXII., Book I.

242. *Is it possible that two* Quadrilateral *figures shall be Equiangular, and yet not similar?*



Let $ABCD$ be any quadrilateral whatever.

Take any (·) a, draw $ab \parallel AB$, and $ad \parallel AD$. Also make ad to ab any ratio unequal to the ratio $AD : AB$.

Then draw $dc \parallel DC$ and $bc \parallel BC$, thus forming the quadrilateral $abcd$ having its sides $\parallel$ those of $ABCD$ each to each.

Join Aa and produce it to K. Then

$$\angle Kad = \text{interior } \angle KAD,$$
$$\text{and } \angle Kab = \angle KAB.$$
$$\therefore \text{ whole } \angle bad = \angle BAD.$$

In the same way it may be shewn that

$$\angle d = \angle D, \ \angle c = \angle C, \ \angle b = \angle B;$$

$\therefore$ $abcd$ is equiangular with $ABCD$.

But by the construction AB, AD, ab, ad are not proportionals;

$\therefore$ $abcd$ and $ABCD$ are not similar.

Q. E. D.

Hence is evident the necessity for both conditions of Definition I. when applied to quadrilaterals. Similarly the same may be shewn for other multilaterals.

Definition II.

243. ***When are two Magnitudes reciprocally proportional to two others?***

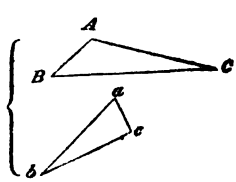
$$\text{If } A : B :: b : a;$$

then

A, a are reciprocally proportional to B, b.

(see art. upon Def. III., Book V.)

244. ***When are $\triangle^s$ reciprocal?***



The $\triangle^s$ ABC, abc, are reciprocal if

$$AB : ab :: ac : AC.$$

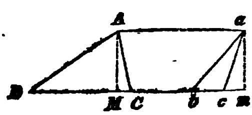
$\square^s$ are reciprocal under the same conditions.

Prop. I.

245. *Is it sufficient to say, "viz. the $\perp$ drawn from the (·) A to BD?"*

It ought to have added "*to BD or BD produced.*"

Prop. I. Cor.

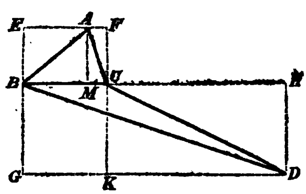


246. Let the △ˢ ABC, abc having = altitudes AM, am be placed so as to have their bases on the same | Bm, and on the same side of it. Join Aa.

∵ am, AM are $\perp$ Bm they are ∥; but they are also =. ∴ Aa is ∥ Bm.

Then taking any equimultiples of the bases BC, bc, and reasoning precisely as in the Proposition, the corollary will be demonstrated.

246. *△ˢ and ▱ˢ upon = Bases are to one another as their Altitudes.*



Let ABC, BCD be two △ˢ having = bases, they shall be as their altitudes.

For let DBC be so placed that its base shall coincide with the = base of ABC, but let their vertices A and D be on different sides of BC. Let AM, DH be the altitudes. From C, B draw $\perp$ˢ to BC, and through A and D draw ∥ˢ to BC, meeting the $\perp$ˢ in E, F, G, K, and forming the ▱ˢ BK, BF.

Then ∵ BK, BF are ▱ˢ of the same altitude BC,

$$\therefore \square BK : \square BF :: BG : BE,$$
$$:: DH : AM.$$

But

$$▱\ BK = 2 \times △\ BCD,$$
$$\text{and } ▱\ BF = 2 \times △\ BAC;$$
$$\therefore 2\ △\ BCD : 2\ △\ BAC :: DH : AM.$$

But magnitudes have the same ratio to one another which their equimultiples have (Prop. XV., Book V.)

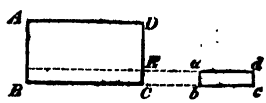
$$\therefore △\ BCD : △\ BAC :: DH : AM.$$

Also ▱ˢ upon = bases being double of △ˢ on same base, and having = altitude, are, in like manner, shewn to be as their altitudes.

Q. E. D.

247. *Required to shew that when neither the Bases nor Altitudes of △ˢ and ▱ˢ are =, they are to one another in the compound Ratio of their Bases and Altitudes; or are to one another as their Bases and Altitudes jointly.*

Let $ABCD$, $abcd$ be any two ▭ˢ of unequal bases, BC, bc, and of unequal altitudes, AB, ab.



$$▭\ ABCD : ▭\ abcd :: AB \times BC : ab \times bc.$$

Place ac so that the base bc may be in the same | with the base BC, and let both ▭ˢ be on the same side of BC. Produce da so as to cut off from BD the ▭ EB.

$$\because ▱^s\ EB,\ ac \text{ have = altitudes},$$
$$\text{and } BE \text{ and } BD \text{ have the same altitude } EC;$$
$$\therefore ac : BE :: bc : BC,$$
$$\text{and } BE : BD :: CE : CD;$$
$$\therefore ▭\ ac : ▭\ AC :: bc \times CE : BC \times CD.$$
$$\text{But } CE = ab,\ CD = AB;$$
$$\therefore ▭\ ac : ▭\ AC :: ab \times bc : AB \times BC.$$

Hence all ▱ˢ are to one another in the compound ratio of their bases and altitudes; ∵ they are = to the ▭ˢ of the same bases and altitudes.

Hence also all △s are to one another in the compound ratio of their bases and altitudes; for they are half the ▭s of the same base and altitude (see also Prop. XXIII., Book VI.)

248. *By the above it may be proved* Geometrically, *that if $S \propto T$ when V is given, and $S \propto V$ when T is given; when neither is given* $S \propto T \times V$ (see Wood's Alg. art. 209).

For, in the above figure, let

AB denote the magnitude T,
BC .... V,
and the area AC .... S;

Also let

ab .... t,
bc .... v,
and the area ac .... s:

Then if area $EB = S'$, we have (Prop. I., Book VI.),

$S : S' :: CD : CE :: AB : ab :: T : t$ (V being given).

Also,

$S' : s :: BC : bc :: V : v$ (T being given).

$$\therefore S : s :: T \times V : t \times v;$$
$$\text{or } S \propto T \times V.$$

Q. E. D.

249. *What is meant by $S \propto T$?*

It implies briefly a proportion of which $S : T$ constitute one of the = ratios, the other being supposed to exist, although not expressed. If $s : t$ be the other ratio, then the proportion, expressed fully, is

$$S : T :: s : t;$$

which is denoted more concisely by

$$S \propto T.$$

When proportions are thus expressed, they are called

VARIATIONS;

and if

$$A \propto B,$$

it means that A "*varies*" as B; that is, if a and b be the unexpressed terms of the proportion, it means, that whilst A changes from A to a, B must so change its magnitude to b, that

$A : B :: a : b$ (see Wood's Alg. art. "*Variations*").

Prop. XXIII.

250. In the three last articles we have purposely deviated from strict Geometry, in order the more forcibly to shew the necessity of the extreme caution the student ought to observe in making his deductions; and also thence to take occasion to dilate, whilst the subject is yet impressed upon the mind, upon the subject of Compound Ratio generally.

In addition to what was advanced (art. upon Def. XI., Book V.), the following observations may possibly clear up the doctrine of Compound Ratio.

Although Euclid, in the Second Book, and other places, frequently compares ▭s with each other, and speaks of them as being "contained" by two |s, yet does he never consider them as the product arising from the multiplication of one | by another. In like manner, a □ is not considered by Euclid as the product of two = multipliers, but only as the area of a plane figure, contained by four = |s at ∟s to each other. The idea of Multiplication, as also of Division, does not, in fact, form any part of the first six Books of the Elements.

In multiplying together, therefore, the terms of the ratios, in the preceding investigations, a departure from rigid accuracy was unquestionably committed.

251. *Required to Compound the Ratios* $A : a$, and $B : b$?

By the vulgar, or Arithmetical method,

$$A \cdot B : a \cdot b$$

would be the compound ratio.

But in Geometry we make, as in Prop. XXIII., Book VI.,

$$A : a :: K : L,$$

$$\text{and } B : b :: L : M.$$

$\therefore$ *ex æquo*,

$$K : M$$

is the compound ratio required.

Otherwise.

Make (Prop. XII., Book VI.),

$$A : a :: A : a$$

$$B : b :: a : P.$$

$\therefore A : P$ is the compound ratio.

252. *Required the Ratio compounded of*

$$A : a, B : b, C : c, D : d, \&c.$$

$$A : a :: A : a$$

Make $B : b :: a : P$ (Prop. XII., Book VI.),

$C : c :: P : Q$ (same)

$D : d :: Q : R$ (same)

&c.

Whence

$$A : R$$

is the compound ratio required.

253. *What is the Ratio compounded of any number of Ratios all* $= A : 1$?

$$A : 1 :: A : 1$$

Make $A : 1 :: 1 : P$ (Prop. XII.)

$$A : 1 :: P : Q$$

$$A : 1 :: Q : R$$

&c.

If X be the last new fourth proportional, the compound ratio required is

$$A : X.$$

254. *How do we hence acquire the notion of Logarithms?*

The logarithm of a ratio is the measure of the composition of that ratio; that is, it is the number which denotes how many = ratios must be compounded to constitute that ratio. Thus (see 253)

$$\text{log. } (A : R) \text{ is } 4;$$

$\because$ 4 is the number of ratios $= A : 1$ necessary to compound $A : R$.

In like manner,

$$\text{log. } (A : P) \text{ is } 2.$$

BOOK XI.

255. This Book treats of |ˢ and rectilinear figures situated in different planes; also of solid ∠ˢ and of solids.

At this University it is not usual to read more than the first 21 Propositions.

DEFINITIONS.

Definition IV.

256. *This Definition is not so briefly conceived as the perpendicularity of planes will admit. It were better thus:*

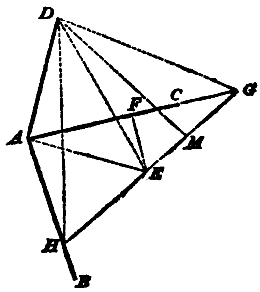
A plane is $\perp$ to a plane, when any | drawn in one of the planes perpendicularly to the common section, is $\perp$ to the other plane.

PROPOSITIONS.

Prop. IV.

257. Although Euclid's treatment of this theorem is exceedingly perspicuous to the experienced in long processes, yet as the demonstration consists of many parts, the Proposition generally proves irksome and difficult to beginners. For this reason we subjoin

Legendre's Proof.



Let AD be $\perp$ to each of the $|^s$ AB, AC at their point of intersection C; it is $\perp$ to the plane CAB.

For draw any other $|$ from A as AE, from E draw $EF \parallel AB$ meeting AC in F; produce AC, and make $FG = AF$. Join GE, and produce it to meet AB in H. Join DH, DE, DG.

$$\because EF \text{ is } \parallel AH, \text{ and } AF = FG,$$
$$\therefore HE = EG \text{ (Book VI.)}$$

Hence, drawing $DM \perp HG$,

$$HD^2 = DE^2 + HE^2 + 2HE \times EM \text{ (Prop. XII., Book II.)}$$
$$DG^2 = DE^2 + GE^2 - 2GE \times EM \text{ (Prop. XIII., Book II.)}$$

$\therefore$ adding these $=^s$,

$$HD^2 + DG^2 = 2DE^2 + 2HE^2.$$

Similarly,

$$HA^2 + AG^2 = 2AE^2 + 2HE^2.$$

But $\because \angle^s$ DAH, DAG are ∟^{s},

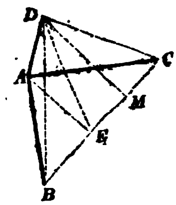
$$\therefore HD^2 = AD^2 + AH^2,$$
$$\text{and } DG^2 = AD^2 + AG^2.$$
$$\therefore HD^2 + DG^2 = 2AD^2 + AG^2 + AH^2.$$
$$\therefore 2DE^2 + 2HE^2 = 2AD^2 + 2AE^2 + 2HE^2.$$
$$\therefore 2DE^2 = 2AD^2 + 2AE^2.$$
$$\therefore DE^2 = AD^2 + AE^2.$$

$\therefore \angle DAE$ is a ∟;

and in the same way it may be proved that DA is $\perp$ to any other $|$ drawn from A in the plane CAB, &c.

Q. E. D.

ANOTHER PROOF, BY THE AUTHOR.



258. Let DA be $\perp$ to AC and AB at A. It is $\perp$ plane CAB.

From A draw any other | AE; from E draw $BC \perp AE$ and meeting AC and BC in B and C, and join DB, DE, DC.

$\because$ the $\angle$s DAB, DAC, AEB, AEC are $\llcorner$s,

$$\therefore DB^2 = DA^2 + AB^2 = DA^2 + AE^2 + EB^2.$$
$$DC^2 = DA^2 + AC^2 = DA^2 + AE^2 + EC^2.$$

Again, if DE is not $\perp$ BC, draw $DM \perp BC$. Then,

$DB^2 = DE^2 + BE^2 + 2BE \times EM$ (Prop. XII., Book II.)

and $DC^2 = DE^2 + CE^2 - 2CE \times EM$.

$$\therefore DA^2 + AE^2 + EB^2 = DE^2 + BE^2 + 2BE \times EM,$$
$$\text{and } DA^2 + AE^2 + EC^2 = DE^2 + CE^2 - 2CE \times EM.$$

Hence

$$2BE \times EM + 2CE \times EM = 0,$$

which is absurd.

$$\therefore DE \text{ is } \perp BC.$$

Hence

$$DA^2 + AE^2 + EB^2 = DE^2 + BE^2.$$
$$\therefore DA^2 + AE^2 = DE^2.$$
$$\therefore \angle DAE \text{ is a } \llcorner, \&c.$$

Q. E. D.

This reasoning, although conducted by symbols, is yet strictly Geometrical.

259. From the above *ex absurdo* result, it follows immediately that

If from the $\angle$ A of the $\triangle$ ABC, AE is let fall $\perp$ upon the opposite side BC, and AD is $\perp$ the other sides at their intersection A, and DE be joined; then DE is also $\perp$ BC.

This is of great use in several Astronomical enquiries;

such as finding the inclinations of the Planetary Orbits to the Ecliptic, and to one another.

For instance, if DCB be the plane of the orbit,
and ABC that of the ecliptic,
then BC, their intersection, is the line of the nodes.

Also ED and EA being each $\perp$ BC, and in the respective planes (by Def. VI., Book XI.), the $\angle$ DEA is the inclination of the orbit to the ecliptic.

260. Hence is also deduced, that

If from any (·) (D) in a plane (DCB) a $\perp$ (DA) be drawn to another plane (CAB); and from the foot of the $\perp$ (A), a $\perp$ (AE) be drawn to the common intersection of the planes (BC), and the extremities of the $\perp$ˢ be joined (D, E); then the | joining the extremities (DE) shall be $\perp$ the intersection, and the $\angle$ contained by it and the other $\perp$ to the intersection ($\angle$ DEA) shall be the inclination of the two planes.

Prop. VI.

261. *Why does Euclid make $DE = AB$?*

By so doing, the demonstration is made to hinge on Prop. IV., Book I., using the △ˢ ABD, BDE; and on Prop. VIII., Book I., with respect to the △ˢ ABE, ADE.

But by using Propp. XLVII., XLVIII., Book I., the demonstration is rendered more direct and simple. Thus,

Let AB, CD be at ∟ˢ to the same plane; they are ∥ to one another.

Let them meet the plane in B, D. Join BD, and in the plane draw $DE \perp BD$, and of any length. Join AE, BE, AD.

$\because AB$ is $\perp$ plane, ABE, ABD are ∟ˢ;

$\therefore AD^2 = AB^2 + BD^2$,

to each add ED^2.

$$\therefore AD^2 + ED^2 = AB^2 + BD^2 + ED^2.$$

But $BD^2 + ED^2 = BE^2$ ($\because \angle BDE$ is a $\llcorner$).

$$\therefore AD^2 + ED^2 = AB^2 + BE^2$$
$$= AE^2.$$

$\therefore$ (Prop. XLVIII., Book I.),

$\angle ADE$ is a $\llcorner$.

$\therefore$ &c. as in Euclid.

Q. E. D.

Prop. VIII.

262. This may be demonstrated more simply, by shewing (as in 261) that

ED is $\perp$ AD.

But by construction,

ED is $\perp$ BD.

$\therefore$ ED is $\perp$ plane CDB, and $\therefore$ $\perp$ CD.

But CD is $\parallel$ AB, and $\angle ABD = \llcorner$.

$\therefore$ $\angle CDB$ is a $\llcorner$.

$\therefore$ CD is $\perp$ BD and DE at their intersection.

$\therefore$ CD is $\perp$ plane BDE.

Q. E. D.

Prop. XIX.

263. Hence,

If three planes, cutting one another, be every two $\perp$ to the third, or, which is the same, be $\perp$ to one another; then the intersections of every two shall be $\perp$ to one another.

Let the planes AB, BC, ADC be all three $\perp$ to one another; also let BD be the intersection of AB, BC; AD that of AB, ADC; and DC that of BC, ADC; the $\angle$' ADB, CDB, ADC are $\llcorner$'.

For $\because$ BC, AB are $\perp$ to plane ADC,

$\therefore$ their intersection BD is $\perp$ to plane ADC (Prop. XIX.)

But AD, DC are in that plane;

$\therefore \angle^s\ BDA$, BDC are $\llcorner^s$.

Again,

$\because AB$, ADC are $\perp$ plane BC;

$\therefore$ their intersection AD is $\perp$ plane BC,

and $\therefore \perp\ |^s\ DB$, DC;

$\therefore \angle^s\ BDA$, ADC are $\llcorner^s$.

$\therefore$ &c.

Q. E. D.

APPENDIX I.

THEORY OF CO-ORDINATES;

OR,

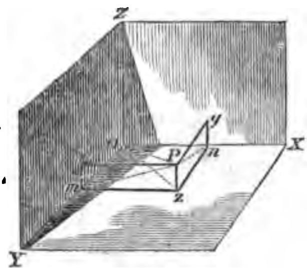
AN INTRODUCTION

TO

ANALYTICAL GEOMETRY.

THEORY

RECTANGULAR CO-ORDINATES.



264. If three planes ZOX, ZOY, YOX are $\perp$ to one another, then their intersections OX, OY, OZ are $\perp$ to one another (see 263).

Planes thus posited are called

Co-ordinate Planes.

265. The intersections of a system of co-ordinate planes (such as OX, OY, OZ) are denominated

Axes of Co-ordinates.

266. The $(\cdot)$ which is common to the three axes of co-ordinates is the

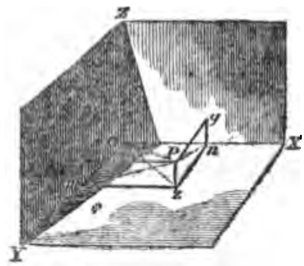
Origin of Co-ordinates.

267. If from any $(\cdot)$ in space (as P), $\|^s$ to the axes of co-ordinates be drawn to the co-ordinate planes (as Px, Py, Pz), these are called the

Co-ordinates of the Point.

268. If the co-ordinate planes be supposed of indefinite extent on both sides of their intersections, they will form eight equal solid $\angle^s$, each formed by three $\llcorner^s$, viz. each = to the solid $\angle$ contained by the plane $\llcorner^s$ YOX, YOZ, XOZ.

In the diagram the $(\cdot)$ P is placed in this latter solid $\angle$, but a similar representation would have sufficed were it situated in any one of the other seven solid $\angle^s$.



269. $\because$ OX is $\perp$ plane ZOY,

and Px is $\parallel$ OX (by Def.),

$\therefore$ Px is $\perp$ plane ZOY.

Similarly it may be shewn, that

Py is $\perp$ plane ZOX,

and that

Pz is $\perp$ plane XOY.

Whence

A co-ordinate of a (·) to any one of the three co-ordinate axes, is $\perp$ to the plane passing through the other two.

270. From z draw $zm \parallel OX$, and join xm; then,

$\because$ zm is $\parallel$ OX, and OX is $\perp$ plane ZOY,

$\therefore$ mz is $\perp$ plane ZOY, and $\therefore$ $\perp$ mx, and $\perp$ OY.

$\therefore$ mz is $\parallel$ OX.

But Px is $\parallel$ OX;

$\therefore$ Px is $\parallel$ mz (Prop. IX., Book XI.)

Hence

xm and Pz are in the same plane with Px and mz (Prop. VII.)

Now $\because$ mz and xP are $\perp$ plane ZOY,

$\therefore$ $\angle^s$ Pxm, zmx are $\llcorner^s$.

Also Pzm is a $\llcorner$.

$\therefore$ Pz is $\parallel$ xm.

But Px is $\parallel$ zm.

$\therefore$ $Pxmz$ is a $\square$,

and $\therefore$ $mz = Px$.

Also, $xm = Pz$.

Similarly, if zn be drawn $\parallel$ OY, and yn be joined, it may be shewn that

$zn = Py$,

and $yn = Pz$.

Hence, then, it appears, that

$$zP,\ zm,\ zn$$

are = to the co-ordinates of the (·) P.

271. *Given the three Co-ordinates of a* (·), *referred to given Co-ordinate axes, to find it.*

From the axes OX, OY cut off On, Om = to the co-ordinates which are $\parallel$ to those axes. From n draw $nz \parallel OY$, and from m draw $mz \parallel OX$, and meeting nz in z. Lastly, from z draw $zP \parallel OZ$, and make zP = to the third co-ordinate; then P is the (·) required.

Since P may be placed in any one of the eight solid $\angle^s$ about O, the problem, unless the signs of the co-ordinates are given, admits of seven other solutions.

(1.) On may be measured from O along XO produced,
Om and zP as in figure.

(2.) On may be taken along XO produced,
Om along YO produced, and zP as in diagram.

(3.) On as in fig.
Om along YO produced, and zP as in fig.

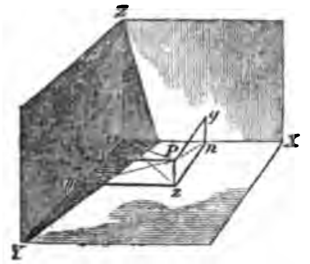
(4.) On, Om, as in fig.
$zP \parallel ZO$ produced.

(5.) On along XO produced,
Om as in fig.
and $zP \parallel ZO$ produced.

(6.) On as in fig.
Om along YO produced,
and $zP \parallel ZO$ produced.

(7.) On along XO produced,
Om along YO produced,
$ZP \parallel ZO$ produced.

272. To distinguish these several cases, Geometers have agreed, from the analogy subsisting between negative quantities



in Algebra, and linear quantities thus measured, to call those co-ordinates which are to be measured along the axes *produced*,

NEGATIVE CO-ORDINATES;

whilst those which are measured along the axes which are *not produced*, are considered

POSITIVE CO-ORDINATES.

Hence denoting, by

$$a, \ b, \ c,$$

the lengths of the co-ordinates of a (·), which are $\|$ to the axes OX, OY, OZ, the position of the (·) will vary through all the eight solid $\angle$ˢ about O, with the changes that may be supposed to take place between a, b, c relatively to *positive* and *negative*.

If, for instance, a be negative, whilst b and c are positive; or if

$$-\ a, \ b, \ c$$

be the values of the co-ordinates of the (·), the only use of the negative sign is to indicate that the co-ordinate $\|$ to the axis OX must be measured along OX produced; and so on for all cases.

THEOREM.

273. *The* □ *described upon the* | *which joins any* (·), *and the origin of Co-ordinates, is* = *to the sum of the* □ˢ *described upon the Co-ordinates of that* (·).

Let P be the (·), O the origin of co-ordinates, &c. From P draw Px, Py, Pz $\|$ to the co-ordinate axes OX, OY, OZ respectively; join OP, Oz; from x and y

draw xm, $yn \parallel OZ$, meeting the axes in m and n, and join nz, mz.

Then (see 270)

$$\because \angle^s\ zmO,\ znO,\ PzO \text{ are } ⅃^s;$$

$\therefore$ (Prop. XLVII., Book I.),

$$PO^2 = Pz^2 + Oz^2,$$

$$\text{and } Oz^2 = Om^2 + mz^2;$$

$$\therefore PO^2 = Om^2 + mz^2 + Pz^2.$$

$$\text{But } mz = Px, \text{ and } Om = Py;$$

$$\therefore PO^2 = Px^2 + Py^2 + Pz^2.$$

Q. E. D.

Or a, b, c being the measured lengths of these co-ordinates,

$$PO^2 = a^2 + b^2 + c^2.$$

THEOREM.

274. *Twice the □ described upon the | joining a (·) and the Origin of Co-ordinates, is = to the sum of the □ˢ described upon the ⊥ˢ drawn from that (·) upon the axes.*

The last figure being retained, join Pm; then (263)

Pm is $\perp$ to the axis OY. Similarly, Pn is $\perp$ OX. And if a $\perp$ be drawn from P to OZ, it will evidently $= Oz$.

Hence (XLVII., Book I.),

$$OP^2 = Oz^2 + Pz^2,$$

$$OP^2 = Pm^2 + Om^2,$$

$$OP^2 = Pn^2 + On^2;$$

$$\therefore 3.OP^2 = Oz^2 + Pm^2 + Pn^2 + Pz^2 + Om^2 + On^2.$$

$$\text{But } Pz^2 + Om^2 + On^2 = OP^2;$$

$$\therefore 2.OP^2 = Oz^2 + Pm^2 + Pn^2.*$$

Q. E. D.

---

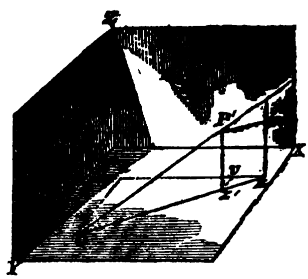
275. * *If the ∠ˢ contained by the | joining any (·) P with the origin, and the several axes of co-ordinates OX, OY, OZ be α, β, γ respectively; then*

276. *Given the Co-ordinates of two (·)ˢ to determine the | passing through them.*

Find each of the (·)ˢ as in (271). Join them; and the | thus found is the one required.

For only one | can pass through two given (·)ˢ (ax. X., Book I.)

277. *Given the Co-ordinates of two (·)ˢ of a | to find those of the (·)ˢ in which it will meet the Co-ordinate Planes.*



Let the co-ordinates of P be

$$a, b, c;$$

those of P' be

$$a', b', c'.$$

Let c and c' meet the plane YOX in z, z'; join z, z'. Then, ∵ Pz, $P'z'$ are $\perp$ to the same plane, they are $\parallel$ (VI., Book XI.); and ∵ Pz, $P'z'$ are $\parallel$;

∴ PP', zz' are in the same plane (VII., Book XI.) Unless ∴ they are $\parallel$, they will meet when produced. Let them meet in Q; from z draw $zu \parallel OX$; from Q and z' draw Qu, and $z'v \parallel OY$; and from P' draw $P'm \parallel Qz$ and meeting Pz in m.

$$\left.\begin{array}{r}\cos.^2\alpha + \cos.^2\beta + \cos.^2\gamma = 1 \\ \text{and } \sin.^2\alpha + \sin.^2\beta + \sin.^2\gamma = 2.\end{array}\right\}$$

For $On = OP.\cos.\alpha$, $Om = OP.\cos.\beta$, $Pz = OP.\cos.\gamma$.

But (274)

$$OP^2 = On^2 + Om^2 + Pz^2 = OP^2\cos.^2\alpha + OP^2\cos.^2\beta + OP^2\cos.^2\gamma;$$

$$\therefore \cos.^2\alpha + \cos.^2\beta + \cos.^2\gamma = 1 \;\ldots\ldots\ldots\ldots (a).$$

Again, $Oz = OP\sin.\gamma$, $Pm = OP\sin.\beta$, $Pn = OP\sin.\alpha$;

$$\therefore (275)\; 2 \times OP^2 = OP^2\sin.^2\gamma + OP^2\sin.^2\beta + OP^2\sin.^2\alpha;$$

$$\therefore \sin.^2\alpha + \sin.^2\beta + \sin.^2\gamma = 2 \;\ldots\ldots\ldots\ldots (b).$$

Calling x, y, the co-ordinates of $Q \parallel OX$ and OY respectively, it is evident, that

$$Qu = y \sim b, \quad z'v = b' \sim b, \quad Pm = c \sim c'.$$

But $\because$ $P'z$ is $\parallel$ Pz, and Qu is $\parallel$ $z'v$, $\therefore$ (Prop. II., Book VI.),

$$Qu : z'v :: Qz : z'z :: Qz : P'm :: zP : mP;$$

$$\therefore y \sim b : b' \sim b :: c : c \sim c';$$

which gives y.

Again,

$$uz : vz :: Qz : z'z :: zP : mP :: c : c \sim c'.$$

But

$$uz = x \sim a, \quad vz = a \sim a';$$

$$\left.\begin{aligned} \therefore x \sim a : a' \sim \quad &:: c : c' \sim c, \\ \text{and } y \sim b : b' \sim b &:: c : c' \sim c; \end{aligned}\right\}$$

which give the co-ordinates x, y, as required for the intersection of the | with the plane YOX. In the same way may proportions be found, involving the co-ordinates of its intersections with the planes ZOX, ZOY.

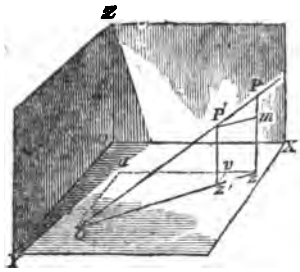
Q. E. I.

Although the above investigation is analytical in appearance, and in the resulting expression, yet none but geometrical reasoning is to be found in it. The whole of this Appendix being designed to bring the student to the very confines of Analytical Geometry, without actually invading it, would often, perhaps, be thought, but for this caution, in the latter predicament.

Theorem.

278. *The □ of the | joining two (·)^s in space, is = to the sum of the □^s of the differences of those Co-ordinates which are $\parallel$ to the same axis.*

Let P, P' be any two (·)^s in space, whose co-ordinates



suppose a, b, c; a', b', c'. Then, making use of the last figure and construction,

$$PP'^2 = P'm^2 + Pm^2.$$

But $P'm^2 = z'z^2 = z'v^2 + zv^2$.

$$\therefore PP'^2 = z'v^2 + zv^2 + Pm^2.$$

But $z'v = b \sim b'$, $zv = a \sim a'$,

$$Pm = c \sim c'.$$

$$\therefore PP'^2 = (a \sim a')^2 + (b \sim b')^2 + (c \sim c')^2.$$

Q. E. D.

Hence is *found the distance of two* $(\cdot)^s$ *in space.*

Also, *the Resultant of any number of forces, resolved into others* $\parallel$ *to three co-ordinate axes,*

&c. &c.

279. *To find the inclination of a* $|$ *in space to each of the Co-ordinate Planes.*

Let PP' be the $|$. Produce it to meet the plane YOX in Q; from P draw $Pz \perp$ plane, and join Qz; then (Def. V., Book XI.) the $\angle\ PQz$ is the inclination required.

In the same manner, the inclinations of PP' to the other co-ordinate planes may be found.*

---

* *To find the trigonometrical expression for the inclinations in terms of the* co-ordinates *of the* $(\cdot)^s$ *which determine the* $|$.

Let a, b, c be the co-ordinates of P,
a', b', c' those of P'.

Then retaining the former construction,

$$Pm = P'm \times \text{tan. } PP'm.$$

But $Pm = c \sim c'$,

$$P'm = zz' = \sqrt{(z'v^2 + zv^2)} = \sqrt{[(b \sim b')^2 + (a \sim a')^2]},$$

and $\angle PP'm = \angle\ PQz$.

$$\therefore c \sim c' = \sqrt{[(a \sim a')^2 + (b \sim b')^2]} \times \text{tan. inclin.}$$

$$\therefore \text{tan. inclin.} = \frac{c \sim c'}{\sqrt{[(a \sim a')^2 + (b \sim b')^2]}} \quad \ldots\ldots\ldots\ldots (c).$$

Otherwise,

$$Pm = PP' \text{ sin. inclin.}$$

280. *Three $(\cdot)^s$ of a Plane not in the same $|$ being given, the Plane is given or determined in position; or, which is the same, through three $(\cdot)^s$ not in the same $|$ there cannot pass more than one Plane.*

For if the $(\cdot)^s$ be joined, the $|^s$ joining them will form a $\triangle$; and $\therefore$ (Prop. II., Book XI.), the three $(\cdot)^s$ are in the same plane.

281. *Given the Co-ordinates of three $(\cdot)^s$ not in the same $|$ of a Plane, to construct it.*

Find each of the $(\cdot)^s$ (271), join them, and the plane of the resulting $\triangle$ will be the plane required.

282. *Given the Co-ordinates of three $(\cdot)^s$ in the Axes of Co-ordinates where the Plane meets the Axes, to find a relation between these Co-ordinates and those of any other $(\cdot)$ in the plane whatever.*

$$\therefore \text{sin. inclin.} = \frac{c \sim c'}{PP'} = \frac{c \sim c'}{\sqrt{[(a \sim a')^2 + (b \sim b')^2 + (c \sim c')^2]}} \ldots\ldots (d).$$

This latter expression indicates, that, if A, B, C denote the inclination of PP' to the planes ZOY, ZOX, XOY respectively, we have

$$\text{sin. } A = \frac{a \sim a'}{PP'}.$$

$$\text{sin. } B = \frac{b \sim b'}{PP'}.$$

$$\text{sin. } C = \frac{c \sim c'}{PP'}.$$

$$\therefore \text{sin.}^2 A + \text{sin.}^2 B + \text{sin.}^2 C = \frac{(a \sim a)^2 + (b \sim b')^2 + (c \sim c')^2}{PP'^2}.$$

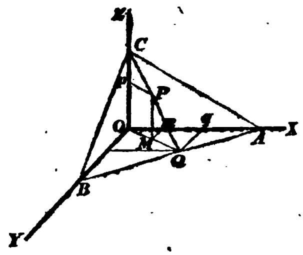
$$\text{But } PP'^2 = (a \sim a')^2 + (b \sim b')^2 + (c \sim c')^2.$$

$$\therefore \text{sin.}^2 A + \text{sin.}^2 B + \text{sin.}^2 C = 1 \ldots\ldots\ldots\ldots (e).$$

This result will be more simply obtained, though not so directly, by drawing from Q $||^s$ to the three co-ordinate axes OX, OY, OZ. The inclinations of PQ to these $||^s$ will be the complements of its respective inclinations to the planes; and by note (page 126),

$$\text{cos.}^2\left(\frac{\pi}{2} - A\right) + \text{cos.}^2\left(\frac{\pi}{2} - B\right) + \text{cos.}^2\left(\frac{\pi}{2} - C\right) = 1.$$

$\therefore$ &c.



Let the co-ordinates of A, B, C, the three given (·)ˢ of the plane, be respectively

$$a, o, o;\ o, b, o;\ o, o, c,$$

viz. $AO = a$, $BO = b$, $OC = c$.

Take any other (·) P in the plane ABC; join CP, and produce it to Q; join OQ, and draw $PM \parallel OC$, Mm, Qq, each $\parallel OB$.

Then PM, Mm, mO are the co-ordinates of P, which denote respectively by z, y, x.

Now by similar $\triangle$ˢ PCp, QCO; and OMm, OQq,

$$pC : OC :: pP : OQ :: OM : OQ :: Om : Oq.$$

$$\text{But } pC = c - z,\ OC = c,\ Om = x.$$

$$\therefore\ c - z : c :: x : Oq.$$

Again, by similar $\triangle$ˢ QqA, BOA,

$$BO : OA :: Qq : qA.$$

$$\text{But } BO = b,\ OA = a, \text{ and } qA = a - Oq.$$

$$\therefore\ b : a :: Qq : a - Oq.$$

$$\text{Also, } Om : Mm :: Oq : Qq,$$

$$\text{or } x : y :: Oq : Qq.$$

$\therefore$ *ex æquo*,

$$bx : ay :: Oq : a - Oq.$$

$\therefore$ *componendo*,

$$bx : bx + ay :: Oq : a.$$

$$\text{But } c - z : c :: x : Oq.$$

$\therefore$ *ex æquo*,

$$bx \times (c - z) : c(bx + ay) :: x : a.$$

$$\therefore\ bx(c - z) : x :: c(bx + ay) : a.$$

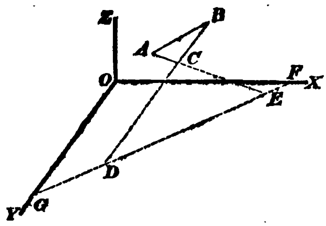
$$\therefore\ b(c - z) : 1 :: c(bx + ay) : a,$$

the relation required.*

---

* If this proportion be converted into an equation, we get

283. *Given three* $(\cdot)^s$ *of a Plane, not in the same* $|$, *to find the* $(\cdot)^s$ *in which it cuts the Axes of Co-ordinates.*



Let A, B, C be the given $(\cdot)^s$, and let ACB be the $\triangle$ formed by their junction. Produce any two of them AC, BC to meet the plane YOX in D and E; join DE and produce it to meet the axes in F and G; then F and G are two of the $(\cdot)^s$ required.

$\because$ the $|^s$ AE, BD intersect in C, they are in the same plane (Prop. II., Book XI.), $\therefore$ by same Proposition, the $|^s$ DC, CE, ED are in the same plane. But AC, CB, BA are in the given plane; $\therefore$ DE is in the given plane; and $\therefore$ F and G, which are in DE, are in the given plane. $\therefore$ F and G are two of the required $(\cdot)^s$.

In the same manner, if any two of the $|^s$ AB, BC, CA be produced to meet either of the planes ZOX, ZOY, may the third intersection be found.

Q. E. I.

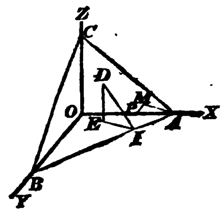
284. *To find the Intersection of a Plane, in which three* $(\cdot)^s$, *not in the same* $|$, *are given, with each of the Co-ordinate Planes.*

By the last Proposition it appears that GF is in the plane ABC, or the given plane, and it is also in the plane YOX; it is, $\therefore$ their intersection. In the same manner may be found the intersections of the plane ABC with ZOX, ZOY.

Q. E. I.

$$abc - abz = bcx + acy,$$
$$\text{or } bcx + acy + abz = abc;$$
and $\therefore$ dividing by abc,
$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1,$$
the analytical equation to a plane.

285. *To find the Inclination of a Plane to each of the Co-ordinate Axes.*



By Def. V., Book XI., "the inclination of a | to a plane is the acute ∠ contained by that |, and another drawn from the (·) in which the first line meets the plane, to the (·) in which a ⊥ to the plane, drawn from any (·) of the first line above the plane, meets the same plane."

Hence, if from any (·) P in the axis OA a ⊥ be drawn to the plane ABC, and A (where OX meets the plane), M be joined, the ∠ MAP is the inclination of OX to the plane ABC; that is, the ∠ MAP is the inclination of the plane ABC to the axis OX. In the same manner may be found the inclinations of the plane ABC to the other axes.

Q. E. I.

286. *To find the Inclination of a Plane to each of the Co-ordinate Planes.*

From any (·) I in the common section AB of the plane ABC with the co-ordinate plane YOX, draw ID in the plane ABC ⊥ AB and IE in the plane YOX ⊥ AB; then (Def. VI., Book XI.) the ∠ DIE is the inclination of the plane ABC to the plane YOX.

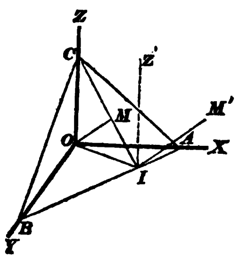
In the same manner may be found the other inclinations.

Q. E. I.

THEOREM.

287. *The Inclination of a Plane to any of the three Co-ordinate Planes, together with the Inclination of that Plane to the Co-ordinate Axis which is ⊥ to the Co-ordinate Plane, is = to a* ∟.

Let ABC be the given plane, BA its intersection with the co-ordinate plane YOX.



From O, the origin of co-ordinates, draw $OI \perp AB$, and C being the intersection of the plane with the axis OZ, which is $\perp$ plane YOX, join CI.

Again, from O draw $OM \perp CI$; and from I draw $IZ' \perp$ plane YOX.

Then $\because$ IZ' and OZ are both $\perp$ plane YOX; they are $\parallel$, and $\therefore$ in the same plane (this is not proved in Euclid, see Prop. VII., Book XI.)

$\therefore$ IO, IC, IZ' are in the same plane (Prop. VII., Book XI.);

$\therefore$ BI is $\perp$ IO, IZ', and $\therefore$ is $\perp$ IC (Prop. IV., Book XI.);

$\therefore$ from I are drawn IC, IO $\perp$ the intersection BA;

the $\angle$ CIO is $\therefore$ the inclination of the planes ABC, YOX.

Again, in the plane CIO draw $IM' \parallel OM$; then the alternate $\angle$ MIM' = alt. $\angle$ OMI = $\llcorner$.

But $\because$ BI is $\perp$ IO and IM, it is also $\perp$ IM' (Prop. IV., Book XI.),

$$\therefore \angle M'IB = \llcorner;$$

$\therefore$ $M'I$ is $\perp$ both IB and IC, and $\therefore$ $\perp$ plane ABC.

But OM is $\parallel$ IM';

$\therefore$ (Prop. VIII., Book XI.) OM is $\perp$ plane ABC.

Now $\because$ OM is $\perp$ plane ABC, and C is the $(\cdot)$ where the $|$ OC meets the plane, and CM are joined; $\therefore$ (Def. V., Book XI.), the $\angle$ OCM is the inclination of the plane ABC to OC.

But $\angle$ OIC is the inclination of the plane ABC to the plane YOX; and $\because$ $\angle$ $COI = \llcorner$,

$$\therefore \angle OCI + \angle CIO = \llcorner.$$

$\therefore$ Inclination of a plane ABC to the co-ordinate plane

YOX, together with its inclination to that co-ordinate axis which is $\perp$ to the co-ordinate plane, equals a $\llcorner$.

In the same way may the proof be extended to the other co-ordinate planes, and the axes which are respectively $\perp$ to them.*

Q. E. D.

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* *Given the co-ordinates of the* (·)* *where the plane meets the three axes of co-ordinates, to find the Inclinations of the plane to the co-ordinate planes.*

Let A, B, C be the (·)* where the plane meets the axis; and suppose the co-ordinates of these (·)* to be

$$a,\ b, \text{ and } c.$$

$$\text{Then } \because OC = OI.\tan. OIC,$$

$$\therefore \tan. OIC = \frac{OC}{OI}.$$

$$\text{But by similar } \triangle^s\ OBI,\ AOB,$$

$$OB : OI :: BA : AO.$$

Or

$$b : OI :: \sqrt{(a^2 + b^2)} : a,$$

$$\text{and } OC = c,$$

$$\therefore \tan. OIC = \frac{c\sqrt{(a^2+b^2)}}{ab};$$

$$\text{or Inclin. of the plane to } YOX = \tan.^{-1} \frac{c\sqrt{(a^2+b^2)}}{ab}.$$

Similarly,

$$\text{Inclin. to plane } ZOX = \tan.^{-1} \frac{b\sqrt{(a^2+c^2)}}{ac},$$

$$\text{and Inclin. to plane } ZOY = \tan.^{-1} \frac{a\sqrt{(b^2+c^2)}}{bc}.$$

APPENDIX II.

A SKELETON

OF THE

TWO BOOKS

OF

APOLLONIUS PERGÆUS

ON TANGENCIES,

AS RESTORED BY

VIETA AND GHETALDUS.

ON TANGENCIES.

THE general Problem of Tangencies (see Pappus's Preface to his Seventh Book, or Halley's Translation), may be reduced to

288. *Of* $(\cdot)^s$, $|^s$ *and* $\odot^s$; *any three whatever being given, to describe a* $\odot$ *which shall pass through the given* $(\cdot)^s$ *and touch the given lines.*

Of this there are ten cases; viz. the data may consist of

three $(\cdot)^s$; *or three* $|^s$;
or two $(\cdot)^s$ *and a* $|$; *or two* $|^s$ *and a* $(\cdot)$;
or two $(\cdot)^s$ *and a* $\odot$; *or two* $\odot^s$ *and a point*;
or two $\odot^s$ *and a* $|$; *or a* $(\cdot)$, *a* $|$, *and a* $\odot$;
or two $|^s$ *and a* $\odot$; *or three* $\odot^s$.

The Fourth Book of Euclid resolves the first case; for to draw a $\odot$ passing three $(\cdot)^s$ which are not in the same $|$ is the same as to circumscribe a $\odot$ about a $\triangle$; and the problem of drawing a $\odot$ touching three $|^s$ which are not $\|$, and which $\therefore$ must meet and form a $\triangle$, is the same as describing a $\odot$ which shall be inscribed a $\triangle$. It must be observed, however (which is not done by Pappus or Halley), that of the latter problem there are four solutions, in one of which the $\odot$ will touch the given lines within the $\triangle$, and in the other three the $\odot$ falls without the $\triangle$.

The six following cases are contained in the "*Liber Primus, De Tactionibus;*" and the last two alone form, with their several cases, arising from the different positions of the lines, the "*Liber Secundus, De Tactionibus.*"

The Tangencies are preceded by a number of Problems,

involving conditions for a construction, which may be enunciated as one, thus:

289. *Of* $(\cdot)^s$, $|^s$, *and* $\odot^s$, *any two whatever being given to describe a* $\odot$ *of given Magnitude, which shall pass through the given* $(\cdot)$ *or* $(\cdot)^s$, *and touch, if possible, the given lines.*

This contains six distinct problems, viz.

Given,

two $(\cdot)^s$; *or two* $|^s$; *or two* $\odot^s$;

or a $(\cdot)$ *and* $|$; *or a* $(\cdot)$ *and* $\odot$; *or a* $|$ *and* $\odot$,

to describe a $\odot$ *which shall pass through the* $(\cdot)^s$ *and touch the lines.*

290. *Through two given* $(\cdot)^s$ A, B *to describe a* $\odot$ *whose radius shall* $=$ *a given line* Z.

Limit. $2Z$ must not be $<$ than the distance AB.

Construction. With centres A and B, and distance Z, describe two arcs cutting one another in E (which they must do by the limit), and E will be the centre required.

[There are two solutions to this Problem; for the $\odot^s$ will cut in two $(\cdot)^s$.—W.]

291. *Given in position two* $|^s$ AB, CD, *required to draw a* $\odot$ *touching them, and having the given radius* Z.

Case 1. Suppose AB and CD to be $\|$.

Limit. $2Z$ must be the distance of the lines, and then the construction is evident.

Case 2. Let AB and CD be inclined to each other, and let them meet in E;

bisect $\angle$ BED by EH;

through E draw $EF \perp ED$ and $= Z$;

through F draw $FG \parallel EH$, meeting ED in G;

and through G draw $GH \parallel EF$.

With centre H and distance $= HG$ describe a $\odot$;

It shall touch the given lines. (Prove this.)

Again, let HI be drawn from $H \perp AB$;
then $HI = HG$ (Prove it);
$\therefore \odot$ also touches AB,
and HG or HI = given line Z (Prove it).

292. *Given two* $\odot^s$ *whose centres are* A *and* B, *required to draw another with radius* Z *touching the two given ones.*

This Problem has various cases, according to the different positions of the given $\odot^s$ and the manner of describing the $\odot$ required; but there are six principal ones, to the conditions of which the rest are subject.

Case 1. Let the $\odot$ to be described be required to be touched *externally* by the given $\odot^s$.

Limit. Then the given diameter $2Z$ must not be < than the segment of the line joining the centres of the given $\odot^s$ which is intercepted between their *convex* $\odot^{ces}$.

Case 2. Let the $\odot$ to be described be required to be touched *internally* by the given $\odot^s$.

Limitation. Then its diameter must not be < than the right line, which, drawn through the centres of the given $\odot^s$, is contained between their *concave* $\odot^{ces}$.

Case 3. Let the $\odot$ to be described be required to be touched *externally* by one of the given $\odot^s$, and *internally* by the other.

Limitation. Then its diameter must not be given < than the segment of the $|$, joining the centres of the given $\odot^s$, which is intercepted between the *convex* $\odot^{ce}$ of one, and the *concave* $\odot^{ce}$ of the other.

Case 4. Let one of the given $\odot^s$ include the other, and let it be required that the $\odot$ to be described be touched *externally* by them both.

Limitation. Then its diameter must not be given > than the > segment of the $|$ joining the centres of the given $\odot^s$,

which is intercepted between the *concave* $\odot^{ce}$ of one, and the *convex* $\odot^{ce}$ of the other; nor $<$ than the less segment.

Case 5. Let one of the given $\odot^{s}$ include the other; required that the $\odot$ to be described be touched *externally* by one of the given $\odot^{s}$, but *internally* by the other.

Limitation. Then its diameter must not be given $>$ than the greater segment of the right line, joining the centres of the given $\odot^{s}$ which is intercepted between the two *concave* $\odot^{ces}$ of the said $\odot^{s}$, nor $<$ than the less segment.

Case 6. Let the given $\odot^{s}$ cut each other, and let it be required that the $\odot$ to be described, and to be touched by them both, shall also be included in each of them.

Limitation. Then its diameter must not be given $>$ than the segment of the $|$, joining the centres of the given $\odot^{s}$, intercepted by their *concave* $\odot^{ces}$, which lies in the space common to both the given $\odot^{s}$.

There may also be three other cases of this Problem, when the given $\odot^{s}$ cut each other; but $\because$ they are similar to cases 1, 2, and 4, already considered, and subject to the same limitations (except that which is similar to the first, which has no limitation), they are here omitted; as are also those cases where the given $\odot^{s}$ touch each other; $\because$ they do not differ from the preceding.

General Solution.

Join the given centres A and B;
and, when necessary by the case,
Let AB be produced to meet the given $\odot^{ces}$ in C and D;
Let CI and DH be taken $= Z$.
With centre A and distance $= AI$, describe $\odot$.
With centre B and distance BH, describe $\odot$.
These $\odot^{s}$ will cut or touch each other (by the limitation).
Let the $(\cdot)$ of concourse be E;

From E draw $|$ EAF, cutting $\odot$ A in F,
and $|$ EBG, cutting $\odot$ B in G; then
with centre E and distance $= EF$ describe $\odot$ FK.
FK is the $\odot$ required.

Demonstration.

$\because AF = AC$, and $AI = AE$;
$\therefore FE = CI$;
But CI was made $= Z$.
$\therefore FE = Z$.
Again, $\because BD = BG$, and $BH = BE$,
$\therefore DH = EG$.
But $DH = Z$.
$\therefore EG = Z$.

Hence $\odot$ FK, passing through F, will also pass through G, and touch $\odot$' in F and G.

293. *Given the* (·) *A and Right Line BC required to draw a* $\odot$, *passing through A and touching BC, and having Radius = the given line Z.*

Limitation. $2Z$ must not be given $<$ than the $\perp$ let fall from the given (·) A upon the given line BC.

From the (·) A draw $AD \perp BC$,
and in AD take $DE = Z$:
Also through E draw $EF \parallel BC$,
and from A upon EF set off $AF = Z$:
Then, with centre F and distance $= FA$, describe a $\odot$.
This $\odot$ touches BC.
F through F draw $FG \parallel AD$;
then $FGDE$ is a $\square$,
and $FG = DE = Z$, and is $\perp$ BC.

294. *Given the* (·) *A, and a* $\odot$ *whose centre is B, to draw a* $\odot$ *whose radius = Z, passing through the given* (·), *and touching the given* $\odot$.

This Problem has three Cases, each subject to a Limitation.

Case 1. Let the $\odot$ to be described be required to be touched *externally* by the given $\odot$.

Limitation. Then the diameter must not be given $<$ than the segment of the right line, joining the given (·) and the centre of the given $\odot$, which is intercepted between the given (·) and the *convex* $\odot^{ce}$.

Case 2. Let the $\odot$ to be described be touched *externally* by the given $\odot$.

Limitation. Then the diameter must not be given $<$ than the right line which, drawn from the given (·) through the centre of the given $\odot$, is contained between the given (·) and the *concave* $\odot^{ce}$.

Case 3. Let the given (·) lie in the given $\odot$.

Limitation. Then a diameter of the given $\odot$ being drawn through the given (·), it is divided into two segments by the said (·), and the diameter of the $\odot$ required must not be given $>$ than the greater of them, nor $<$ than the less.

General Solution.

Join AB;

in AB take $CF = Z$;

With centre A and distance $= Z$, describe an arc, and with centre B and distance $= BF$, draw another, cutting the former in D;

then, with centre D and distance DA draw a $\odot$;

it shall touch the given $\odot$.

For

DB being drawn meeting the given $\odot$ in E,

$$BC = BE.$$

$$\therefore\ CF = ED = Z.$$

295. *Given the* | *BC, and the* $\odot$ *whose centre is A, to*

draw a ⊙ whose Radius = Z, and which shall touch the | and also the given ⊙.

This Problem has three Cases, each of which is subject to a Limitation.

Case 1. Let the ⊙ to be described be touched *externally* by the given ⊙.

Limitation. Then the diameter of the ⊙ required must not be given < than the segment of a line, drawn from the centre of the given ⊙, ⊥ to the given line, which is intercepted between the said line and the *convex* ⊙^{ce}.

Case 2. Let the ⊙ to be described be required to be touched *internally* by the given ⊙.

Limitation. Then the given line must not be in the given ⊙, neither must the diameter of the ⊙ required be given < than that portion of the ⊥, drawn from the centre of the given ⊙ to the given line, which is intercepted between the said line and the *concave* ⊙^{ce}.

Case 3. Let the ⊙ to be described be required to be both touched by and included in the given ⊙.

Limitation. Then the given | must be in the given ⊙, and when a diameter of this given ⊙ is drawn cutting the given line at ∟^{s}, the diameter of the ⊙ required must not be given > than the greater segment.

General Solution.

From A draw $AB \perp BC$ and cutting ⊙ in D,
and in this ⊥ take BG and DF each = Z.
Through G draw $GE \parallel BC$;
and with centre A and distance AF describe an arc
which, by the limitations, will meet GE in E.
Join AE, and, if necessary, produce it to the ⊙ in H.
Then, with centre E and distance EH, describe a ⊙.
It shall be the ⊙ required.

For it evidently will touch the given ⊙.

Also, $\because$ $AD = AH$, and $AF = AE$,

$\therefore$ $DF (= Z) = HE$.

Draw $EC \perp BC$.

Then $GBCE$ is a ▱.

$\therefore$ $EC = GB = Z$.

$\therefore$ $\because$ ECB is a ∟, the ⊙ will also touch the given line BC.

The Tangencies.

296. Given two (·)ˢ A, B, and a | EF, to find the centre of the ⊙ passing through the given (·)ˢ, and touching the given line.

This admits of two answers.

297. Given a (·) A and two |ˢ BC and DE, to draw a ⊙ passing through the given (·), and touching both the given lines.

298. Given a ⊙ in magnitude and position, and two |ˢ in position, to draw a ⊙ touching all three of them.

Two Cases.

299. Given a (·) A, a |, and a ⊙ in magnitude and position, to describe a ⊙ passing through the (·) and touching both the | and the ⊙.

Two Cases.

300. Given two ⊙ˢ in magnitude and position and a |, to draw a ⊙ touching all three.

Four Cases.

301. Given two (·)ˢ and a ⊙, to describe another ⊙ passing through the (·)ˢ and touching the given ⊙.

Several Cases.

302. Given two ⊙ˢ and a (·), to draw a ⊙ touching the given ⊙ˢ, and passing through the given (·).

Three Cases.

303. Given three ⊙ˢ, to draw a fourth touching all three of them.

Four Cases.

Those who wish to see these Tangencies fully demonstrated may consult a small 4to., by Lawson, entitled "The Two Books of Apollonius Pergæus concerning Tangencies, &c." (See also farther on in this Work beginning with art. 337, page 161).

APPENDIX III.

MISCELLANEOUS DEDUCTIONS;

WITH OCCASIONAL

HINTS AND SOLUTIONS.

MISCELLANEOUS DEDUCTIONS.

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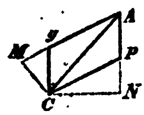
304. *If, from the extremity of the diagonal of a* ▱, $\perp^s$ *be drawn to the sides produced, the* $\triangle^s$ *contained by the sides not produced, the parts of the sides produced cut off by the* $\perp^s$, *and the* $\perp^s$, *shall be similar* (see Whewell's "Mech." art. 28).



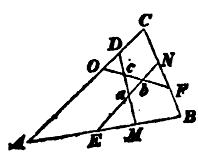
Let pq be a ▱.

From C let CM be drawn $\perp$ Aq produced in M, and $CN \perp Ap$ produced in N. The $\triangle^s$ CpN, CqM shall be similar.

$$\because \text{ the opposite } \angle^s \text{ of a ▱ are } =,$$
$$\therefore \angle\, ApC = \angle\, AqC.$$
$$\text{But } \angle\, ApC + CpN = \angle\, AqC + CqM.$$
$$\text{Take away the } = \angle^s\, ApC,\ AqC,$$
$$\text{and remaining } \angle\, CpN = \text{ remaining } \angle\, CqM.$$
$$\text{Also } \angle^s \text{ at } M \text{ and } N \text{ are } ∟^s.$$
$$\therefore \text{ remaining } \angle^s\, qCM,\ pCN \text{ are equal.}$$
$$\therefore \triangle^s\, CpN,\ CqM \text{ are equiangular, and } \therefore \text{ similar.}$$

Q. E. D.

305. *If* $|^s$ *be drawn making* $= \angle^s$ *with the three sides towards the same parts of any* $\triangle$, *they will form a* $\triangle$ *similar to the given* $\triangle$.—Whewell's "Mech." art. 31.



Let MD, NE, OF make with the sides AC, AB, BC the $\angle^s$ MDA, NEB, OFC all $=$, and form the $\triangle$ abc. abc is similar to ABC.

For in the △s AMD, EMa,
$\angle\, aEM = \angle\, MDA$ by supposition,
and $\angle\, AMD$ is common;
$\therefore$ remaining $\angle\, A$ = remaining $\angle\, EaM$
$= \angle\, cab$.

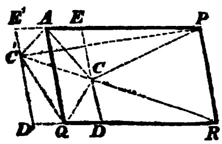
Similarly it may be shewn, that
$\angle\, B = \angle\, abc$
and $\angle\, C \therefore = \angle\, acb$;
$\therefore$ △ abc is similar to △ ABC.

Q. E. D.

306. *Hence, if* ⊥s *be drawn to the three sides of a* △, *they will form a* △ *similar to the given* △ (see Wood's "Mech.")

307. *If from the four* ∠s *of any* ▱, |s *be drawn to a given* (·), *half the* ▱ *shall = the sum or difference of the* △s *whose bases are opposite sides of the* ▱, *and common vertex the given* (·); *according as the* (·) *is between or not between the bases.* Whewell, art. 32.

For if C fall between, through C draw $DE \parallel PR$, &c. as in fig.



Then

$$\triangle\, ACQ = \tfrac{1}{2} \text{▱}\, EQ,$$
$$\triangle\, PCR = \tfrac{1}{2} \text{▱}\, ER;$$
$$\therefore \triangle\, ACQ + \triangle\, PCR = \tfrac{1}{2} \text{▱}\, AR.$$

Again,

$$\triangle\, PC'R = \tfrac{1}{2} \text{▱}\, PD',$$
$$\triangle\, AC'Q = \tfrac{1}{2} \text{▱}\, AD';$$
$$\therefore \triangle\, PC'R - \triangle\, AC'Q = \tfrac{1}{2} \text{▱}\, AR.$$

Q. E. D.

308. *If from the four* ∠s *of a* ▱, |s *be drawn to any* (·), *the* △ *formed by the diagonal and the two* |s *drawn to its extremities, is = to the difference or sum of the* △s

formed by the other two |* *and one of the former, according as the* (·) *is within the* ▭ *or not, that is, supposing* AR *joined,*

$$\left.\begin{array}{l} \triangle ACR = \triangle ACP - \triangle ACQ, \\ \text{and } \triangle AC'R = \triangle AC'Q + \triangle AC'P. \end{array}\right\}$$

For $\triangle ACQ + \triangle PCR = \frac{1}{2} \square AR$ (by 307)

$$= \triangle PAR$$
$$= PACQR - \triangle ACR$$
$$= \triangle ACP + \triangle PCR - \triangle ACR.$$

$$\therefore \triangle ACR = \triangle ACP - \triangle ACQ.$$

Again,

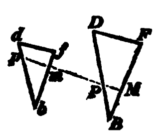
$$\triangle PC'R - \triangle AC'Q = \tfrac{1}{2} \square AR$$
$$= \triangle APR$$
$$= PAC'R - \triangle AC'R$$
$$= \triangle AC'P + \triangle PC'R - \triangle AC'R;$$

$$\therefore \triangle AC'R = \triangle AC'P + \triangle AC'Q.$$

Q. E. D.

309. *If from any* (·) *a* ⊥ *be drawn to a plane, and from the foot of the* ⊥ *a* ⊥ *be drawn to* | *in the plane, and a* | *be drawn from the given* (·) *to the* (·) *where the last* ⊥ *meets the* |; *then the* | *last drawn is* ⊥ *to the* | *in the plane.* See Whewell's "Mech." Prob. VII., art. 41, and page 133 of this Work.

310. *Three* |* *drawn* ∥ *to the three sides of a* △ *make a* △ *similar to the given* △ (see Newton's "Principia," Lemma XI., Case 3).



Let bd be ∥ BD,
df be ∥ DF,
and fb be ∥ FB;
the △ dbf is similar to the △ DBF.

For if the | pM be drawn ∥ to DF and cutting the sides in P, M, p, m, or them produced, it will also be ∥ df, ∵ df is ∥ DF.

and $A'B' : AD :: CA' : CA$ (Euc., Book VI).

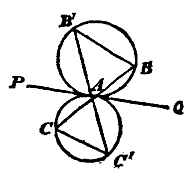
$\therefore$ compounding the ratios

$$1 : 1 :: CA' \times \angle ACD : CA \times \angle ACB.$$

$$\therefore \angle ACB : \angle ACD \text{ or } A'CB' :: A'C : AC.$$

Q. E. D.

315. *If from the* $(\cdot)$ *of contact of two* $\odot^s$, *two* $|^s$ *be drawn terminated both ways by the* $\odot^{ces}$, *and the extremities in each* $\odot^{ce}$ *be joined, two* $\triangle^s$ *will be formed similar to one another* (see Newton, Prop. L.)



From the $(\cdot)$ of contact A let BC, $B'C'$, any two $|^s$ be drawn terminated by the $\odot^s$ in B, B'; C, C'; the $\triangle^s$ ABB', ACC' are similar.

For draw PQ touching the $\odot$ BAB' in A. Then it also touches the $\odot$ CAC', $\because$ the centres of the $\odot^s$ and A lie in the same $|$, and $\therefore$ PQ is $\perp$ to both diameters in the $(\cdot)$ A.

But $\because$ PQ touches the $\odot$ BAB',

$$\therefore \angle BAQ = \angle B';$$

and $\because$ PQ touches the $\odot$ CAC',

$$\therefore \angle PAC = \angle C'.$$

But $\angle PAC = \angle BAQ$.

$$\therefore \angle B' = C',$$

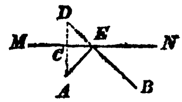
and the vertical $\angle^s$ at A are $=$.

$\therefore$ $\triangle^s$ ABB', ACC' are similar.

Q. E. D.

316. *From two given* $(\cdot)^s$ *on the same side of a given* $|$, *to draw* $|^s$ *making* $=$ $\angle^s$ *at the same* $(\cdot)$ *and on the same side of the given* $|$. (Mechanics and Optics.)

Let A, B be the given (·)ˢ, MN the given |. It is required to draw from A and B two |ˢ AE, EB such that $\angle\ AEM = \angle\ BEN$.



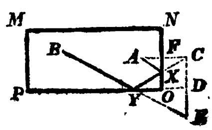
From A draw $AC \perp MN$, and produce it, making $CD = AC$, and join DB, cutting MN or MN produced in E, and join AE; AE, EB are the |ˢ required.

$$\because \text{ in } \triangle^s\ ACE,\ DCE,$$
$$AC,\ CE = DC,\ CE \text{ each to each,}$$
$$\text{and } \angle\ ACE = \angle\ DCE;$$
$$\therefore \angle\ AEC = \angle\ DEC.$$
$$\text{But } \angle\ BEN = DEC.$$
$$\therefore \angle\ AEC = \angle\ BEN.$$

Q. E. D.

This Proposition is of great use in the theories of *Collision of Bodies*, and of the *Reflection and Refraction of Light*. The *games of Billiards and Tennis* also involve it.

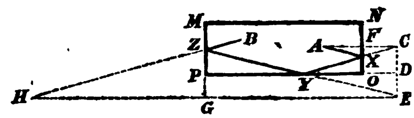
317. *If MO be a* ▭, *required to draw the rectilinear path AXYB from the given* (·) *A to the given* (·) *B, such that AX, XY shall be equally inclined to NO and XY, YB equally inclined to PO.*



Draw $AF \perp NO$, and produce AF, making $FC = AF$. From C draw $CD \perp PO$, meeting PO produced in D; produce CD, making $DE = CD$. Join BE, cutting PO in Y. Join YC, cutting NO in X, and join AX; then $AXYB$ is the path required.

The proof is like the last.

N. B. If at *Billiards* it were required to make the ball A strike the two cushions NO, PO, and finally strike the ball B, the (·) X would be that where it would be first reflected.



318. *Required to find the path AXYZB from the (·) A to the (·) B, such that AX, XY may be equally inclined to NO; that XY, YZ may be equally inclined to PO, and YZ, ZB to PM.*

From A draw $AF \perp NO$, and make $FC = AF$.

Draw $CD \perp PO$, meeting PO in D, and produce CD, making $DE = DC$.

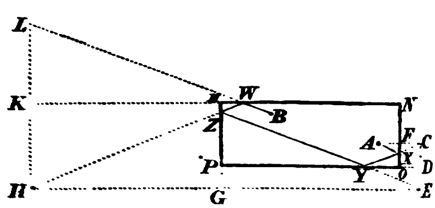
Draw $EG \perp MP$, meeting MP produced in G, and produce EG, making $GH = GE$.

Join HB, cutting MP in Z; ZE, cutting PO in Y; YC, cutting NO in X; and AX; then $AXYZB$ is the path required.

The proof is the same as that of 316.

N. B. If the billiard ball A be struck in the direction AX, after striking the three cushions NO, OP, PM, it will finally strike the ball B.

319. *Required to find the path AXYZWB from the (·) A to the (·) B, such that AX, XY may be equally inclined to NO; XY, YZ to PO; YZ, ZW to PM, and ZW, WB to MN.*



Having found the (·) H as in the last problem, draw $HK \perp MN$, meeting MN produced in K, produce HK, and make $KL = HK$.

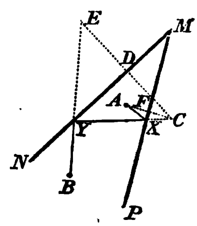
Join BL, cutting MN in W; WH cutting PM in Z; ZE cutting PO in Y; YC cutting NO in X; and AX; then $AXYZWB$ is the path required.

The proof is the same as that of 316.

N. B. The billiard ball being struck in the direction AX, after being reflected from all the cushions of the table, would finally strike the ball B.

It must be remarked, that, if A and B have such positions that any of the (·)^s X, Y, Z, W fall without the ▭, or be in the sides produced, then the above feats at billiards become impossible.

320. *Required to draw the path AXYB from the* (·) *A to the* (·) *B such that AX, XY may be equally inclined to MP and XY, YB equally inclined to MN, which is or is not* $\parallel$ *to MP.*



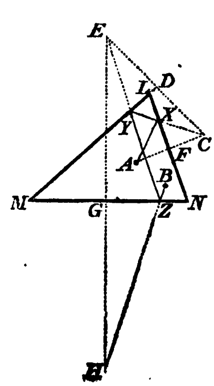
Draw $AC \perp MP$, making $FC = AF$; then draw $CE \perp NM$, making $DE = CD$; and join EB, cutting NM in Y.

Join YC, cutting MP in X. Finally, join AX, and $AXYB$ is the path required.

The proof is evident from 316.

Upon constructions like this depends the theory of the Reflection of *Looking-glasses* or *Plane Mirrors*, the *Kaleidoscope*, &c.

321. *Required to draw the path AXYZB from A to B, such that AX, YX may be equally inclined to LN; XY, YZ to LM and YZ, ZB to MN, the remaining side of the* △ *LMN.*



Draw $AC \perp LN$, making $FC = FA$; draw $CE \perp LM$, making $DE = DC$; draw $EH \perp MN$, making $GH = EG$; and join

HB, cutting MN in Z,
EZ, cutting LM in Y,
YC, cutting LN in X.

Finally, join AX; $AXYZB$ is the path required.
Prove it by 316.

322. In the same way may a path be drawn from one given (·) to another so as to be equally inclined by its pairs of branches to each of the sides of any polygon, regular or irregular.

323. *Required to draw from a given (·) a rectilinear path meeting all the sides of a Polygon, and returning to the (·), such that each pair of its contiguous sides may be equally inclined to a side or side produced of the Polygon.*

The construction is the same as that of 321 would be if the two (·)ˢ A and B were to coincide.

324. *If the two Diagonals of any Quadrilateral be bisected, and the (·)ˢ of bisection joined, the □ˢ of the Diagonals, together with four times the □ of the line joining the (·)ˢ of bisection, are = to the □ˢ of the four sides of the figure.*

Use Propp. XII. and XIII., Book II.

325. *Parallel |ˢ in a ⊙ intercept = ⊙ᶜᵉˢ.*

Join the opposite extremities of the ||ˢ; so as to get alternate ∠ˢ equal.

Prove the theorem by Prop. XXVI., Book III.

326. *If in the diameter of a ⊙ any two (·)ˢ equidistant from the centre be taken, and |ˢ be drawn from them to any (·) in the ⊙ᶜᵉ, then the □ˢ of these lines taken together shall be = to the □ˢ of lines drawn from them to any other (·) of the ⊙ᶜᵉ.*

Join the centre and (·) taken in the ⊙^{ce}.

Prove by Propp. XII. and XIII., Book II.

327. *To find the inclination of two* |^{s} *without producing them until they meet.*

Take any (·) in one of them, and from it draw a | ∥ to the other; the ∠ thus formed = the ∠ required.

328. *The three* |^{s} *which bisect the* ∠^{s} *of a* △ *intersect each other in the same* (·).

First prove that two of them will intersect; and then making the supposition that the third does not intersect in this (·) of intersection, but falls in a different situation, prove *ex absurdo* that it does not bisect the ∠. Whence it will follow that the supposition was false, and ∴ that all three |^{s} do intersect in the same (·).

In the construction it will be necessary to let fall ⊥^{s} upon the sides from the (·) of intersection.

Otherwise.

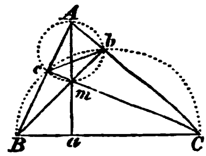
Vary the proof, by joining the third ∠ and the (·) of intersection, and then producing to the opposite side.

329. *The three* |^{s} *which join the* ∠^{s} *of a* △ *and the middle* (·)^{s} *of the opposite sides, intersect in the same* (·).

Two of them will evidently intersect. Join the middle (·)^{s} to which these are drawn. The | thus drawn will be ∥ to a side of the △. Join the third ∠ with this intersection, and produce to the opposite side. Then, by means of similar △^{s}, shew that the third side is bisected.

330. *The three* |^{s} *which are drawn from the* ∠^{s} *of a* △ ⊥ *to the opposite sides all intersect in the same* (·).

Two of them, as *Bb*, *Cc*, evidently cut one another. Join their intersection and the third ∠, and produce to the opposite side, and prove that the | thus drawn is ⊥ to the third side.



∵ Bb, Cc intersect in m, and the $\angle$s b and c are $\llcorner$s,

∴ Am is the diameter of a ⊙ passing through $Abmc$. Also BC is the diameter of a ⊙ passing through $BcbC$.

Hence,

$$\angle\ bBC = \angle\ bcm = \angle\ bAm.$$

$$\text{But } \angle\ Amb = \angle\ Bma.$$

$$\therefore\ \angle\ Bam = \angle\ Abm = \llcorner.$$

∴ the three $\perp$s intersect in the same (·) m.

Some vainly imagine that this does not prove the Proposition, but only that

If any $\angle$ of a △ be joined with the intersection of the $\perp$s drawn from the other two $\angle$s upon the opposite sides, and produced, it will meet the third side at $\llcorner$s.

331. *The two Diagonals of a ▱ bisect each other.*

The segments, together with two of the ∥ sides, make two △s, having two $\angle$s = to two $\angle$s each to each, and a side in each =, viz. sides adjacent to the = $\angle$s.

∴ by Prop. XXVI., Book I., &c.

332. *If a | be drawn from the Vertex of a △ bisecting the Base, all Lines drawn ∥ to the Base, and intercepted by the Sides of the △ will be bisected by the same |.*

Centre of Gravity.

Use Prop. II., Book VI.

333. *If the four sides of a Quadrilateral be bisected, and the (·)s of bisection in the contiguous sides be joined by |s, the Quadrilateral formed by these |s will be a ▱.*

Centre of Gravity.

Use Prop. II., Book VI.

334. *If any (·) be taken in the Radius of a ⊙, and a second (·) in the Radius produced on the same side of the Centre, so that its distance from the Centre may be a third proportional to the distance of the first (·) from the Centre and the Radius; then two |ˢ drawn from these (·)ˢ to any (·) in the ⊙ᶜᵉ will have the same Ratio as that of the two parts into which the | comprised between these (·)ˢ is divided by the ⊙ᶜᵉ.* (Optics.)

From the (·) in the radius produced draw a tangent to the ⊙, join the (·) of contact with the extremities of the diameter, and with the centre. Also let fall from it a ⊥ upon the diameter.

Use Propp. XI., Book VI.; VI., Book VI.; XXXII., Book I.; V., Book I; and III. of Book VI.

335. *All the Diagonals which can be drawn in a regular Pentagon are =, and they will cut each other in mean and extreme Ratio.*

The first is evident.

The second is proved by Propp. XXVIII., Book III.; XXVII., Book III.; XXVII., Book I.; XXXIV., Book I.; V., Book I.; and IV., Book VI.

336. *In a given |, to find a (·) equally distant from two given (·)ˢ.*

Join the given (·)ˢ, and bisect the resulting | by a ⊥ to it produced to meet the given |. The (·) where the ⊥ meets the | is the (·) required.

Prove by Prop. IV., Book I.

CONTACTS.

337. *To draw a | touching two given ⊙ˢ.*

If the ⊙ˢ are =, draw radii ⊥ to the |, joining their

centres; then the | joining the (·)^s thus found in the ⊙^{ces} will be the tangent required.

If they are not =, from the centre of the greater draw a ⊙ whose radius is the difference between those of the given ⊙^{s}; then from the centre of the less draw a tangent to the new ⊙; join the (·) of contact with the centre, and produce it to meet the ⊙^{ce} of the greater ⊙. Draw a radius of the less ⊙ ∥ to this radius of the greater ⊙, and the two extremities of the radii will be the (·)^s of contact required.

Of each Case there will be four Solutions.

338. *If to every two of three unequal ⊙^{s} there be drawn a pair of common Tangents, each pair will intersect, and the three (·)^s of intersection will lie all in the same* | .

Join the centres in pairs, by |^s, which will meet the pairs of tangents in their intersections. Join also the several (·)^s of contact with the centres of the ⊙^{s} touched, and from the centre of that ⊙ which is the mean in magnitude, draw a | ∥ to the | joining two of the vertices, and meeting the | which joins the centres of the other two ⊙^{s}; then join the third vertex with the line joining the other two vertices; and prove, by means of an *ex æquo*, Propp. XV., Book V.; VI., Book VI.; XXIX, Book I.; and XIV., Book I., that the |^s joining the three vertices make ∠^{s} at the middle vertex with the | passing through the centres which together are = to two ∟^{s}, &c.

339. *To draw a ⊙ touching two given* |^s.

If the |^s are ∥, draw a ⊥ to both |^s, and bisect it. The (·) of bisection will be the centre of the ⊙ required.

If the |^s are not ∥, produce them to meet at an ∠. Bisect the ∠ by a |, in which any (·) being taken, will be equally distant from the given |^s, and ∴ the centre of a ⊙ which may be described.

340. *To draw a ⊙ touching two given* |^s, *and passing through a given* (·).

By 339, draw any $\odot$ touching both $|^s$.

Join the given $(\cdot)$ with the intersection of the given $|^s$, by a line which will cut the touching $\odot$ in two $(\cdot)^s$. Join that which is nearest to the given $(\cdot)$ with the centre of the touching $\odot$; then, from the given $(\cdot)$ draw a $|$ $\parallel$ to this radius of the $\odot$, meeting the $|$ produced which joins the intersection of the given $|^s$, and the centre the touching $\odot$. The $(\cdot)$ where these $|^s$ meet is the centre of the $\odot$ required.

341. *Through two given* $(\cdot)^s$, *to describe a* $\odot$ *touching a given* $|$.

Produce the $|$ joining the $(\cdot)^s$ to meet the given $|$ (when these are not $\parallel$), and from the $(\cdot)$ where they intersect take along the given $|$ a mean proportional to its distances from the given $(\cdot)^s$, which will determine the $(\cdot)$ of contact of the required $\odot$ with the given $|$. Finally, draw a $\odot$ passing through this and the given $(\cdot)^s$; this is the $\odot$ required.

There will be two Solutions.

When the $|^s$ are $\parallel$, the problem presents no difficulty.

342. *Through two given* $(\cdot)^s$, *to describe a* $\odot$ *touching a given* $\odot$.

Describe a $\odot$ passing through the two given $(\cdot)^s$, and *cutting* the given $\odot$ in any two $(\cdot)^s$ whatever; and then produce the lines passing through the given $(\cdot)^s$ and through the two latter $(\cdot)^s$ until they meet. From the $(\cdot)$ where they meet draw a tangent to the given $\odot$; then the $\odot$ which passes through the $(\cdot)$ of contact, and the two given $|^s$ is the $(\cdot)$ required.

This Problem will have two Solutions.

343. *To describe a* $\odot$ *which shall pass through a given* $(\cdot)$, *touch a given* $|$, *and touch a given* $\odot$.

Through the centre of the given $\odot$ draw a $\perp$ to the given $|$, and from the farther extremity of the resulting diameter

draw a | to the given (·). Cut this | into two such parts in a (·) x, that the ▭ by the whole and the part nearest the given ⊙ may be = to that by the whole ⊥ and the diameter of the given ⊙. Finally (by 341), draw a ⊙ passing through the given (·) and the (·) of section, and touching the given |; this will also touch the given ⊙, and ∴ be the ⊙ required.

There are Two Solutions.

344. *Through a given* (·) *C, to describe a* ⊙ *touching two given* ⊙*s whose centres are A and B, the former being the greater.*

Produce the line AB to D so that

$$AD : BD :: \text{radius of } A : \text{radius of } B.$$

Join DC and produce it to L, so that $DC \times DL$ may = the □ of the tangent drawn from D to the ⊙ whose centre is B.

Finally, draw a ⊙ passing through L and C, and touching either of the given ⊙s; it will also touch the other, and be the ⊙ required.

Two Solutions.

345. *To describe a* ⊙ *which shall touch a given* | *and two given* ⊙*s.*

From the centre of the less ⊙ draw a ⊥ upon the given | and produce it, making the part produced = to the radius of the other ⊙; at the extremity of this draw a | ∥ to the given line. Also, with the centre of the less ⊙, and at the distance = the difference between the radii of the given ⊙s describe a ⊙. Finally, draw a ⊙ touching the latter ⊙, passing through the centre of the greater ⊙, and touching the | drawn ∥ to the given |. Concentric with this draw another at the distance of the given |, and it will be the ⊙ required.

Two Solutions.

346. *To describe a $\odot$ touching a given $\odot$ and two given $|^s$.*

$\|$ to the given $|^s$, and distant from them by the radius of the given $\odot$, draw other $|^s$, and through the centre of the given $\odot$ draw a $\odot$ touching the new $|^s$. Concentric with this draw a $\odot$ touching either $|$, and it will also touch the other and the given $\odot$, as required.

Two Solutions.

347. *Required to describe a $\odot$ touching three given $\odot^s$.*

If the $\odot^s$ are all unequal, concentric with the two greatest describe others whose radius shall be the difference between their radius and that of the least given $\odot$; then draw a $\odot$ passing through the centre of the least given $\odot$, and touching the two new ones (344); and, concentric with this, draw a $\odot$ touching any one of the three given $\odot^s$, and it will touch both the others, as required.

Here we have two Solutions, and several cases in each Solution.

Several of these Tangencies are useful in Mechanics, to find the Lines of Quickest Descent, &c.

Loci.

348. *A Locus is that line in which certain lines or $(\cdot)^s$ obtain an indefinite number of positions.*

349. Prop. XXXV., Book I. affords an instance, inasmuch as the $|$, which is $\|$ to the base, is the *Locus* of the sides of the $=$ ▱^{s}.

350. Prop. XXXVI., Book I. gives also two Loci, viz. the $\|$ $|^s$; and the two following Propositions present Loci, the same relatively to the vertices of $=$ △^{s}, as those of the two preceding relatively to $=$ ▱^{s}.

351. Book III., Prop. I. Cor. makes the $\perp$ bisecting a | in a $\odot$ the Locus of the centre of the $\odot$.

352. By Prop. XXI., Book III., we learn, that a segment of a $\odot$ is the Locus of an $\angle$ subtended by the same base.

353. Prop. XXXI., Book III., shews the Locus of a ∟ subtended by the same | , to be a semi-circle.

This subject, although as extensive as infinity itself, is very limited when treated with reference to utility. Indeed, whether interesting or useful, it generally leads to higher Geometry than that of Euclid, viz., the higher Conics, or Transcendental Curves.

Maxima and Minima.

354. *The greatest ▭ that can be formed by the segments of a given line, is that made when the segments are equal.*

Prove it by Prop. V., Book II.

355. Propp. VII. of Book III.; VIII., Book III.; XV., Book III. are examples.

356. *Of all = $\triangle^s$ upon the same base, that has the greatest vertical $\angle$ which is Isosceles.*

Since they are =, they are between the same $\parallel^s$. Draw a $\parallel$ to the base through the vertex, and describe a $\odot$ passing through the extremities of the base, and touching this $\parallel$. That $\triangle$ whose vertex is the (·) of contact will be easily shewn (Prop. XXI., Book III.) to be the maximum $\triangle$, &c.

357. *Of all the $\triangle^s$ upon the same base, and whose vertices are in the same given | , required that whose vertical $\angle$ is a Maximum.*

Through the extremities of the given base describe a $\odot$ which also touches the given Locus of the vertices. The (·) of contact is the $\triangle$ required.

This Prop. resolves many curious questions in Optics relating to the maximum appearance of objects.

For instance, since the apparent magnitude of an object is proportional to the ∠ which it subtends at the eye, if that (·) in the observer's rectilinear Locus were found, at which the ∠, found as above, were the greatest, it would give the position where the object would appear a maximum.

Thus, if a person recede from the steeple of a church, the spire, at a certain (·), would appear longer than at any other distance.

358. *Of all the △ˢ upon the same base, and the Locus of whose vertices is a given ⊙ in the same plane, required to find those two, one of which has a Maximum, and the other a Minimum vertical ∠.*

Through the extremities of the base draw a ⊙ touching with its *convexity* the given ⊙ (see 342). The (·) of contact determines the *maximum* vertical ∠.

Again, draw ⊙ through the same (·)ˢ, but touching with its *concavity* the given ⊙. In this case, the (·) of contact will determine the *minimum* ∠ required.

Many curious, and perhaps, in some instances, useful consequences, would flow from this construction.

For example, with certain modifications, it may enable one to ascertain pretty nearly the best seat for vision in the circular row of boxes at a theatre.

359. It would be easy to extend this theory to other Loci of the spectators. *Required, for instance, of all the △ˢ upon the same base, and the Locus of whose vertices is the surface of a given Sphere, to find those whose vertical ∠ˢ are Maxima or Minima.*

To accomplish this, through the extremities of the given base draw a sphere which shall also touch the given sphere

both externally and internally; that is, with both its convex surface and its concave surface. The former in its various positions, will give a Locus of (·)ˢ of contact, which will each determine a *maximum* ∠, and the latter will give another Locus, the (·)ˢ of which will each determine a *minimum* ∠.

This question, which involves a series of Deductions from Euclid (usually to be found prefacing Treatises on Spherical Trigonometry), the student must defer resolving until he has made himself master of Book XI., and the Theory of the Intersection of Planes with Spheres, &c.

When resolved it will conduct him to many interesting speculations relative to Geometry of Three Dimensions, amongst which one may be mentioned as the most obvious, viz.

360. *Given the absolute length and position of a Comet's tail, referred to the ⊕'s centre (as ascertained by Practical Astronomy), to find those (·)ˢ on the ⊕'s surface, as defined by Latitude and Longitude, at which the tail would appear to be the greatest and least.*

361. *Of all △ˢ on the same Base, and having = vertical ∠ˢ, the Isosceles is the greatest.*

Since the vertical ∠ˢ are =, the vertices lie in the same segment standing on the given base.

362. *Of all △ˢ standing upon the same Base, and on the same side of it, and having = vertical ∠ˢ, that which is Isosceles has the greatest perimeter.*

Use Propp. XXI., Book III.; XX., Book III.; XXXII., Book I.; VI., Book I.; and XV., Book III.

363. *Of all Polygons contained by the same number of sides, and inscribed in the same ⊙, the greatest is that which has all its sides equal.*

Join the farthest extremities of any two contiguous sides, and the resulting isosceles △ will be > than any other that

can be drawn on the same base, and whose vertex is in the same part of the $\odot^{ces}$. Hence the polygon, two of whose sides are $=$, is $>$ all other polygons having the same sides except two and those two unequal.

Again, two other sides of the polygon being changed for other $=$ sides, the polygon becomes still $>$ than before; and proceeding thus continually, the more nearly the polygon becomes equilateral, the $>$ does it become; so that it may be inferred, that ultimately arriving at an equilateral polygon, this polygon is the maximum required.

This is the method of proof usually adopted. For an ample discussion of this and kindred subjects, see the "Maxima and Minima," as treated by Dr. Cresswell.

CONSTRUCTIONS.

364. *Given three $|^s$, every two of which is $>$ than the third, to construct the $\triangle$.*

This is Prop. XXII., Book I.

How happens it in Trigonometry, that when the three sides of a $\triangle$ are given to find the $\angle^s$, the above condition relative to the sums of every two of the sides, and the third side, is not stated?

365. Propp. I., XI., and XII., Book I.; XXIII., Book I.; XXXI., Book I.; XLII., Book I.; XLIV., Book I.; XLV., Book I.; XLVI., Book I.; XIV., Book II.; XVII., Book III.; XXV., Book III.; XXXIII., Book III.; all the Propp. of Book IV.; Propp. XXV., Book VI.; XXVIII., Book VI.; XXIX., Book VI.; XI., Book XI.; XII., Book XI.; are all Constructions.

366. *Given two sides, and the included $\angle$ of a $\triangle$ to construct it.*

Take one of the sides, and at its extremity make an $\angle$ $=$ the given $\angle$; and cut off from the $|$ last drawn a $|$ $=$

the other side. Then complete the △, and it will be that required.

367. *Given two ∠ˢ, together < than two ∟ˢ, and a Side, either adjacent to the given ∠ˢ or else opposite to one of them, to construct the △.*

If the given side is adjacent to the = ∠ˢ, from its extremities describe ∠ˢ = the given ∠ˢ, by |ˢ which will meet and form the required △.

If the given side is opposite to one of the given ∠ˢ, find the third ∠ of the △, and this Case is reduced to the first.

LEGENDRE, in his "*Problêmes relatifs aux deux premiers Livres*" of his Geometry, in resolving this Problem, overlooks the condition that the two given ∠ˢ must be together < than two ∟ˢ.

368. *Given two sides, a, b, of a △ and the ∠ B opposite to b, to construct the △.*

Let any two indefinite |ˢ make the given ∠ *B*; from one of them cut off a part = *a*, and from its extremity *C*, with distance = *b*, describe a ⊙ cutting the other of the indefinite |ˢ (if possible) in the (·) *A*; then *ABC* will be the △ required.

This will be impossible whenever *b* is < than the ⊥ from the extremity of *a* upon the other side containing the given ∠ *B*.

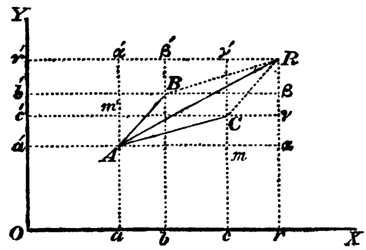
369. *Given the adjacent sides of a ▱, and the ∠ contained by them, to construct the ▱.*

From the extremities of the given sides draw |ˢ ∥ to the given sides, which will meet and complete the ▱.

370. The preceding Problem suggests the following

THEOREM.

If AB, AC be any two $|^s$ intersecting in A, and the $\square$ $ABRC$ be completed (as in 369), and from its four $\angle^s$ $A, B, C, R \perp^s$ be drawn to each of two $|^s$ OX, OY $\perp$ to each other in O, viz. Aa, Aa'; Bb, Bb'; Cc, Cc'; Rr, Rr'; then



$$Rr - Aa = Bb - Aa + Cc - Aa,$$
$$\text{and } Rr' - Aa' = Bb' - Aa' + Cc' - Aa'.$$

For producing $b'B$, $c'C$, $a'A$ to β, γ, and α, it is easily seen that $RB\beta$, CAm are similar, and $= \triangle^s$, and $\therefore$ that

$$R\beta = Cm.$$

$$\therefore Rr - Aa = R\alpha = R\beta + \beta\alpha = Cm + \beta\alpha$$
$$= Bb - Aa + Cc - Aa.$$

Similarly,

$$Rr' - Aa' = R\alpha' = R\beta' + \beta'\alpha' = Cm' + \beta'\alpha'$$
$$= Bb' - Aa' + Cc' - Aa'.$$

Q. E. D.

Hence, if the co-ordinates (see 268)

$$Aa, Bb, Cc \text{ be } b, y, y',$$

and the co-ordinates

$$Aa', Bb', Cc' \text{ be } a, x, x',$$

then

$$\left.\begin{aligned} Rr - Aa &= (y - b) + (y' - b), \\ \text{and } Rr' - Aa' &= (x - a) + (x' - a). \end{aligned}\right\}$$

It also hence appears that (XLVII., Book I.)

$$\begin{aligned} AR^2 &= Ra^2 + Aa^2 \\ &= (Rr - Aa)^2 + (Rr' - Aa')^2 \\ &= \{(y - b) + (y' - b)\}^2 + \{(n - a) + (x' - a)\}^2. \end{aligned}$$

These results mean that the co-ordinates of R are the respective sums of the co-ordinates of B and C; and that the □ of the diagonal R of the ▱ equals the sum of the □ˢ of those sums.

This serves as a Geometrical demonstration of the Theorem of the ▱ of Forces, as deduced from the resolution of the two forces AB, AC resolved ∥ to the axes of co-ordinates OX, OY.

Something of this kind, or else an employment of the Arithmetic of Sines, is necessary for the comprehension of Laplace's deductions from his Theory of the Resolution of Forces. For such students, indeed, as aspire to reading the "Mecanique Celeste" (and all other reading should be but preparatory to this), it will be useful not only to bear in mind this construction, but also to extend it to three dimensions. They will then comprehend Laplace's Corollary as to the Parallelopipedon of Forces: not that Laplace himself makes any use of either, beyond resolving all forces ∥ to co-ordinate axes, and then compounding them into one resultant.

Having thus given a variety of the more useful Deductions, nearly all that occur in questions relative to Natural Philosophy, as well as those connected with the higher branches of Abstract Mathematics, it only remains to make a selection of others which may serve the purpose of exercising the reader's own unassisted ingenuity, and of requiring from him more severe Self-Examinations than he has hitherto been subjected to.

The Lune of Hippocrates.

371. *A Semi-circle described on the Hypothenuse of a*

right-angled △ is = to two Semi-circles described on the other two sides; and if the ½⊙ on the Hypothenuse be drawn so as to meet each of the other ½⊙s in two (·)s, the two Lunes, *or Crescents, thus formed, will together be = to the △.*

This depends upon Prop. II., Book XII.

N. B. This XII. Book need not be read further than Prop. II., as they never require any Proposition beyond that, at this University.

372. *Given the Altitude = to the Base of a △ and its two sides, to construct it.*

373. *If any number of |s be placed in a ⊙, to determine the locus of their intersections.*

374. *Required to divide into any number of = rings a given ⊙.*

Dr. Hutton prided himself much upon being the first to resolve this question; and takes occasion to descant largely upon the merits of his discovery. It was proposed originally thus:

A number of men agreed to buy a grind-stone, for which paying equally, they were to grind down = portions with their scythes or sickles. Of course, the said grind-stone was, in its surface, cylindrical. The question, then, that naturally arose amongst them was, to what depth of the semi-diameter each grinder should grind, in order to lose no part of his scantling. This computation was reserved for Dr. Hutton (see his Miscellaneous Tracts), who did that thing which I imagine will prove no proof of genius to many readers.

375. *Required to multisect a ⊙ by means of Semi-circles.*

This is easy, if you first divide the diameter into the required number of parts.

376. *If a common Tangent be drawn to any number of ⊙s which touch each other internally, and from any (·), as*

a centre, a $\odot$ be described, cutting the others, and from the centre $|^s$ be drawn through the intersections of the $\odot^s$ respectively; the segments of them within each $\odot$ will be equal.

377. *Required the longest stick that can be put up a Chimney, the height of its orifice and other dimensions being given.*

378. *A ladder is just long enough, if properly placed, to reach the parapets of the houses on both sides of a street, required the position of the ladder that it may just effect this, when the houses are not of the same height.*

379. *Given the hypothenuse of a right-angled $\triangle$, and the sum or difference of the base and $\perp$ to construct the $\triangle$.*

380. *Given the base of a $\triangle$, its altitude, and the ▭ under its sides, to construct it.*

There are two Solutions.

381. *To bisect a ▱ by a $|$ drawn from a $(\cdot)$ in one of its sides.*

382. *To bisect a Trapezium by a $|$ from one of its $\angle^s$.*

383. *To bisect a Trapezium by a $|$ from a $(\cdot)$ in one of its sides.*

384. *The $\square^s$ of the diagonals of a ▱ are together $=$ to the $\square^s$ of the four sides.*

385. *Six Sovereigns will exactly surround another, and be each in contact with it and two others.*

386. *To divide a $|$ into two parts, such that the ▭ contained by them may be $=$ to the $\square$ of the difference.*

387. *To find two $|^s$ such that the sum of their $\square^s$ may be $=$ to a given $\square$, and their ▭ $=$ to a given $\angle$.*

388. *To divide a given $\triangle$ into any number of $=$ parts by $|^s$ drawn $\parallel$ to a given $|$.*

389. *Two* $(\cdot)^s$ *being given without a* $\odot$, *to find a* $(\cdot)$ *in the* $\odot^{ce}$, *from which* $|^s$ *drawn to the two given* $(\cdot)^s$ *shall contain the greatest* $\angle$ *possible.*

390. *To inscribe a* $\square$ *in a right-angled isosceles* $\triangle$.

391. *To inscribe a* $\square$ *in the quarter of a* $\odot$.

392. *To inscribe a* $\square$ *in a* $\frac{1}{2}\odot$.

393. *To inscribe a* $\square$ *in a segment of a* $\odot$.

394. *In a* $\triangle$ *to inscribe a* ▱ *similar to a given* ▱.

395. *In a* $\triangle$ *to inscribe a* $\triangle$ *similar to a given* $\triangle$.

396. *In a given pentagon to inscribe a* $\square$.

397. *If the diameter of a* $\frac{1}{2}\odot$ *be divided into any number of parts, and on them semi-circles be described; their* $\odot^{ces}$ *will together be* $=$ *to the* $\odot^{ce}$ *of the given* $\frac{1}{2}\odot$.

398. *To find the Locus of the vertex of a* $\triangle$, *whose base and ratio of the other two sides are given.*

399. *If an* $\triangle$ *be inscribed in a* $\odot$, *and through the angular* $(\cdot)^s$ *another be circumscribed; to determine the ratio which they bear to each other.*

400. *If an* $\triangle$ *be inscribed in a* $\odot$, *and the adjacent arcs cut off by two of its sides be bisected; the line joining the* $(\cdot)^s$ *of bisection will be trisected by the sides.*

401. *To draw an Arithmetic Mean between two given* $|^s$.

402. *To draw a Geometric Mean between two given* $|^s$.

403. *To draw a Harmonic Mean between two given* $|^s$.

404. *Shew that of any two given* $|^s$ *the Geometric Mean is also a Geometric Mean between their Arithmetic and Harmonic Means.*

405. *Required to construct two, three, or more Arithmetic means between two given* $|^s$.

406. *Construct* 3, 7, 15, 31, $2^6 - 1$, *or* 63, $2^7 - 1$, *or* 127, *&c. Geometric Means between two given* $|^s$.

407. *Shew it is impossible to deduce from Euclid's Elements, that is, by means of* $|^s$ *and* $\odot^s$ *only, the construction of* two *Geometric Means between two given* $|^s$; *and thence explain the impossibility of the famous Problems of the* TRISECTION OF AN ANGLE, *and of the* DUPLICATION OF A CUBE.

408. *If any three* $\angle^s$ *of an equilateral Pentagon be* $=$, *the Pentagon is also equiangular.*—Euc., Book XIII., Prop. VII.

409. *If the side of a Hexagon and the side of a Decagon inscribed in the same* $\odot$ *be added together, the whole* $|$ *will be cut in mean and extreme Ratio.*—Euc., Book XIII., Prop. IX.

410. *If a regular Pentagon, a regular Hexagon, and a regular Decagon be inscribed in the same* $\odot$, (*the side of the Pentagon*)^2 $=$ (*side of the Hexagon*)^2 $+$ (*side of the Decagon*)^2.—Euc., Book XIII., Prop. X.

411. *If the Semi-diameter or Radius of a* $\odot$ *be bisected, and a* $\perp$ *be drawn to it from its centre, meeting the* $\odot^{ce}$, *the* $|$ *joining this* $(\cdot)$ *in the* $\odot^{ce}$ *and that of bisection is* $=$ *to the side of the inscribed regular Pentagon.*

412. *The* $\square$ *of the side of an* $\triangle$ *inscribed in a* $\odot$ *is* $=$ *to* 3 *times the* $\square$ *of the Radius of the* $\odot$.—Euc., Book XIII., Prop. XII.

413. *Construct a Tetraedron, and shew that* (*its side*)^2 $= \frac{8}{3}$ (*Radius of the Circumscribing Sphere*)^2.—Euc., Book XIII., Prop. XIII.

414. *Construct an Octaedron, and shew that* (*its side*)^2 $= 2$ (*Radius of the Circumscribing Sphere*)^2.—Euc., Book XIII., Prop. XIV.

415. *Construct a Cube, and demonstrate that* (*its side*)^2 $= \frac{4}{3}$ (*Radius of the Circumscribing Sphere*)^2.—Euc., Book XIII., Prop. XV.

416. *Construct an Icosaedron, and shew that (its side)^2 $= \frac{2}{5}(5 - \sqrt{5}) \times$ (Radius of the Circumscribing Sphere)^2.*—Euc., Prop. XVI., Book XIII.

417. *Construct a Dodecaedron, and shew that (its side)^2 $= \frac{2}{3}(3 - \sqrt{5}) \times$ (Radius of its Circumscribing Sphere)^2.*—Euc., Book XIII., Prop. XVII.

418. *There cannot be more than five regular bodies.*—Euc., Book XIII., Prop. XVII., Schol.

419. *Only three kinds of regular plane figures can fill up the plane space about a (·); viz. six △^{s}, four □^{s}, and three regular Hexagons.*

Why do bees, in the construction of their cells, adopt the hexagonal form?

420. *Only three kinds of regular plane figures can form a solid ∠; viz. 3, 4, or 5 △^{s}, 3 □^{s}, and 3 Pentagons.*

421. *Only one of the five regular bodies can of itself completely fill the space about a (·); viz. eight Cubes.*

422. *The surface of a ⊙ = Radius $\times \frac{⊙^{ce}}{2}$.*

423. *The sector of a ⊙ = Radius $\times \frac{Arc}{2}$.*

424. *Given any number of ⊙^{s}, to find another whose surface shall = the sum of the surfaces of the given ⊙^{s}* (Prop. II., Book XII.)

425. *Given any number of similar plane figures, to find another similar to the given figure, and = in surface to the sum of them.*

426. *Given any number of Spheres, to find another whose volume or solidity shall = the sum of the volumes of the given spheres.*

427. *Given any number of similar Solids, to find one similar to them, and = in volume to the sum of their Volumes.*

428. *Of all Plane Figures of = Periphery, the ⊙ has the greatest surface.*

429. *Of all Plane Figures of = Surface, the ⊙ has the least Periphery.*

430. *For the same Periphery, the □ has a greater Surface than the △; and for the same Surface, the Periphery of a □ is < than that of an △.*

431. *For the same Periphery, a Hexagon has a greater Surface than a Pentagon, a □, or an △; and for the same Surface, it has a less Perimeter than any of them.*

432. *If a □ be inscribed in a ⊙, its Surface* $= 2 \times (\textit{Radius})^2$, *and the Circumscribed □ is double the Inscribed □.*

433. *In a regular Pentagon,* $(\textit{its side})^2 = \frac{5 - \sqrt{5}}{2} \times (\textit{Radius of circumscribing } \odot)^2$.

434. *In a regular Pentagon,*

$$\textit{the Surface} = \tfrac{5}{8}\sqrt{(10 + 2\sqrt{5})}.(\textit{Radius of circumscribing } \odot)^2,$$
$$\textit{or} = \tfrac{1}{4}\sqrt{(25 + 10\sqrt{5})}.(\textit{its side})^2.$$

435. *In a regular Hexagon, the surface* $= \frac{3\sqrt{3}}{2}.(\textit{its side})^2$.

436. *In a regular Octagon,* $(\textit{its side})^2 = (2 - \sqrt{2}) \times (\textit{Radius of its circumscribing } \odot)^2$.

437. *In a regular Octagon,*

$$\textit{the Surface} = (2 + 2\sqrt{2}).(\textit{its side})^2,$$
$$\textit{or} = 2\sqrt{2}.(\textit{Radius of its circumscribing } \odot)^2.$$

438. *In a regular Decagon, its side* $= \frac{\sqrt{5} - 1}{2}(\textit{Radius of its circumscribing } \odot)$.

439. *In a regular Decagon,*

the Surface $= \frac{5}{2}\sqrt{(5+2\sqrt{5})}.(\textit{its side})^2.$

or $= \frac{5}{4}\sqrt{(10-2\sqrt{5})}.(\textit{Radius of its circumscribing} \odot)^2.$

440. *In a regular Dodecagon,*

its side $= \sqrt{(2-\sqrt{3})} \times (\textit{Radius of its circumscribing} \odot),$

or $= \frac{1}{2}(\sqrt{6}-\sqrt{2}) \times (\textit{Radius of its circumscribing} \odot).$

441. *In a regular Dodecagon,*

the Surface $= (6+3\sqrt{3})\,(\textit{its side})^2,$

or $= 3\,(\textit{Radius of its Circumscribing} \odot^2).$

442. *In a regular Quindecagon,*

its side $= \frac{1}{4}\{\sqrt{(10+2\sqrt{5})}-\sqrt{15}+\sqrt{3}\} \times (\textit{Radius of} \odot).$

443. *In a regular Quindecagon,*

the Surface $\frac{15}{16}\{\sqrt{15}+\sqrt{3}-\sqrt{(10-2\sqrt{5})}\} \times (\textit{Radius of its Circumscribing} \odot)^2.$

444. *The Volume of a Parallelopiped = its Base × its Altitude.*

445. *The Volume of a Prism = its Base × its Altitude.*

446. *The Curved Surface of*
a Cylinder = the $\odot^{ce}$ *of its Base × its Altitude,*
or Surface $= 2\pi.r \times a;$
in which π *is the* $\odot^{ce}$ *of the* $\odot$ *whose Diameter is 1 (that is, in which* $\pi = 3{\cdot}14159$ *nearly),* r *the Radius of the Circular Base, and* a *the Altitude.*

447. *The Volume of a Cylinder = Base × its Altitude; or, in symbols,*

$$V = \pi r^2 a.$$

448. *The Volume of a Pyramid = its Base ×* $\frac{1}{3}$ *of its Altitude,*

or $V = \frac{1}{3}$ *Base ×* a.

449. *The Curved Surface of a Cone = the* $\odot^{ce}$ *of its Base ×* $\frac{1}{2}$ *its Side,*

$$\text{or } S = \pi . r . L,$$
$$\text{or } = \pi r . \sqrt{(r^2 + a^2)}.$$

450. *The Volume of a Cone = its Base × ⅓ of its Altitude,*

$$\text{or } V = \frac{\pi r^2 a}{3}.$$

451. *The Curved Surface of a Truncated Cone, the Radii of whose Bases are R and r, and Altitude a, is*

$$S = \pi (R + r) \sqrt{\{(R - r)^2 + a^2\}}.$$

452. *The Volume of a Truncated Cone,*

$$V = \tfrac{1}{3}\pi . a (R^2 + r^2 + Rr).$$

453. *The Surface of a Sphere = its Diameter × ⊙ce of its great ⊙.*

or = 4 × Surface of its great ⊙.

In Symbols,

$$S = 4\pi r^2.$$

454. *The Volume of a Sphere = its Surface × ⅓ of its Radius,*

$$\text{or } V = \tfrac{4}{3}\pi r^3.$$

455. *The Surface of a Spherical Lune, whose Spherical* $\angle$ *is* $A = \frac{A}{2\pi} \times$ *Surface of the Sphere,*

$$\text{or } S = 2Ar^2.$$

456. *The Surface of a Spherical △ whose* $\angle$*s are A, B, C = the excess of the sum of these* $\angle$*s above two* ∟*s × (Radius of the Sphere)²; or*

$$S = (A + B + C - \pi) r^2.$$

457. *The Surface of a Spherical Polygon whose* $\angle$*s are* $A_1, A_2 \ldots A_n$, *is* $S = \{A_1 + A_2 + \ldots A_n - (n - 2) . \pi\} r^2$.

458. *The Surface of any Spherical Zone with one*

Base, or with two = its Altitude × $\odot^{ce}$ *of the Sphere's great* $\odot$,

$$\text{or } S = 2\pi r . a.$$

Hence, with slight addition, may be estimated the extent of surface of the respective Zones of the Earth, Frigid, Temperate and Torrid.

459. *If R and R′ be the radii of two* ∥ *Bases of a Segment of a Sphere, whose Radius is r, the Volume of the Segment is*

$$V = \frac{\pi}{3}\sqrt{\{(r^2 - R^2) - \sqrt{(r^2 - R'^2)}\}}.\{r^2 + R^2 + R'^2 - \sqrt{(r^2 - R^2)(r^2 - R'^2)}\}.$$

Or if a be the altitude of the segment,

$$V = \frac{\pi}{6}a\,(3R^2 + 3R'^2 + a^2) \text{ or } = \pi R^2.\frac{a}{2} + \pi R'^2.\frac{a}{2} + \frac{4\pi}{3}.\left(\frac{a}{2}\right)^3,$$

that is, the Segment = two Cylinders whose Bases are those of the Segment, and common Altitude half that of the Segment + a Sphere whose Radius is half the Altitude of the Segment.

460. *The Surface of a Sphere* = 2 × *Curve Surface of inscribed Equilateral Cylinder,*
and = $\frac{4}{3}$ *of its whole Surface.*

The Surface of a Sphere = Curve Surface of the Circumscribed Cylinder,
and = $\frac{8}{3}$ *Curve Surface of inscribed Equilateral Cone,*
and = $\frac{2}{3}$ *Curve Surface of Circumscribed Equilateral Cone.*

Definition.

An Equilateral Cylinder is that whose altitude = the diameter of its base.

An Equilateral Cone is that whose slant side = diameter of its base.

461. *A Spherical Shell is the space contained between two concentric Spheres, and its volume is ∴ the difference between the volumes of the Spheres; or R and r being the radii of the Spheres, the volume of the Shell is*

$$V = \frac{4\pi}{3}.(R^3 \sim r^3).$$

This comes into use in the Theory of the Earth, and in that of the Tides.

462. The Volume of a Sphere = $\frac{32}{9}$. inscribed Equilateral Cone,
and = $\frac{4}{9}$ circumscribed Equilateral Cone,
and = $\frac{4\sqrt{2}}{3}$ inscribed Equilateral Cylinder,
and = $\frac{2}{3}$ circumscribed Cylinder.

463. For the same surface the Sphere is the most capacious of all bodies; and, for the same volume, the Sphere has the least surface of all bodies.

464. For the same surface and same altitude, that Right Prism whose base is a regular Hexagon, has a > volume than any other of a < number of sides; and, for the same volume, the Hexagonal Prism has a < surface than any other of a < number of sides (see 430, &c.)

465. *The Altitude of a Tetraedron = $\frac{2}{3}$ the diameter of its Circumscribing Sphere.*

466. *The Surface of a Tetraedron* $= \frac{8\sqrt{3}}{3}.R^2$, *in which R is the Radius of its Circumscribing Sphere.*

467. *The Volume of a Tetraedron* $= \frac{8\sqrt{3}}{27}.R^2$.

468. *The Surface of a Hexaedron or Cube* $= 8.R^2$.

469. *The Volume of a Cube* $= \frac{8\sqrt{3}}{9}.R^3$.

470. *The Surface of an Octaedron* $= 4\sqrt{3}.R^2$.

471. *The Volume of an Octaedron* $= \frac{4}{3}R^3$.

472. *The Surface of a Dodecaedron* $= 2.\sqrt{(50-10\sqrt{5})}.R^2$.

473. *The Volume of a Dodecaedron* $= \frac{2}{9}(5\sqrt{3}+\sqrt{15}).R^3$.

474. *The Surface of an Icosaedron* $= (10\sqrt{3}-2\sqrt{15}).R^2$.

475. *The Volume of an Icosaedron* $= \frac{2}{3}\sqrt{(10+2\sqrt{5})}.R^3$.

NUMERICAL APPROXIMATIONS.

476. *Van Ceulen, having estimated the ⊙ce of a ⊙ true to thirty-six decimal places ,ordered it to be engraven on his Sepulchre; but the indefatigable Abraham Sharpe, by a very laborious process, brought it up to seventy-two decimal places, and Lagny took the fruitless pains to proceed in the career, by a different method, as far as 127 decimal places. If π denote the Ratio of the ⊙ce of a ⊙ to its diameter, Lagny's computation gives*

$$\pi = 3{\cdot}141592,653589,793238,462643,383279,502884,197169,$$
$$399375,105820,974944,592307,816406,286208,998628,$$
$$034825,342117,067982,148086,513272,306647,093844,6.$$

It need hardly be observed, that this extreme approximation is not proposed to students in these days, when so many

branches of study, higher, and vastly more important than that of Geometry, demand their attention. It is put down, however, as a curiosity.

To find the value of π true to seven decimal places, which is, for almost all purposes, sufficiently accurate, it suffices, first, to obtain the surface of the □ˢ inscribed in, and circumscribed about a ⊙ whose radius is 1; thence to get those of the octagon; thence those of the regular polygon of 16 sides, and so on. By proceeding Geometrically in this way, these results will be obtained, viz.

Number of Sides.	Surface of Inscribed Polygon.	Surface of Circumscribed Polygon.
4	2	4
8	2·8284271	3·3137085
16	3·0614674	3·1825979
32	3·1214451	3·1517249
64	3·1365485	3·1441184
128	3·1403311	3·1422236
256	3·1412772	3·1417504
512	3·1415138	3·1416321
1024	3·1415729	3·1416025
2048	3·1415877	3·1415951
4096	3·1415914	3·1415933
8192	3·1415923	3·1415928
16384	3·1415925	3·1415927
32768	3·1415926	3·1415926

whence the surface of a ⊙ whose radius is 1, or the ⊙ᶜᵉ of a ⊙ whose diameter is 1, is nearly

3·1415926.

(See Brewster's "Translation of Legendre's Elements of Geometry," page 100.)

Otherwise, true to 6 places of decimals.

Number of Sides.	Perimeter of Inscribed Polygon.	Perimeter of Circumscribed Polygon.
6	6·	6·822033
12	6·211657	6·430781
24	6·265257	6·319320
48	6·278700	6·292173
96	6·282063	6·285430
192	6·282904	6·283747
384	6·283115	6·283327
768	6·283167	6·283221
1536	6·283180	6·283195
3072	6·283184	6·283188
6144	6·283185	6·283186

which gives

$$2\pi = 6{\cdot}283185 \text{ nearly,}$$
$$\text{or } \pi = 3{\cdot}141592 \text{ nearly.}$$

Either of these methods, if continued far enough into decimals, will give the value of π true to any required degree of accuracy.

477. If s denote the side of any Regular Polygon, then the approximate values of their surfaces are respectively as follows, viz.

Surface of $\triangle$ $= 0{\cdot}4330127 \times s^2$.
$\square$ $= s^2$.
Pentagon $= 1{\cdot}7204774 \times s^2$.
Hexagon $= 2{\cdot}5980762 \times s^2$.
Heptagon $= 3{\cdot}6339124 \times s^2$ (not to be investigated by Euclid).
Octagon $= 4{\cdot}8284271 \times s^2$.
Nonagon $= 6{\cdot}1818242 \times s^2$ (not a deduction).
Decagon $= 7{\cdot}6942088 \times s^2$.
Undecagon $= 9{\cdot}3656399 \times s^2$ (not a deduction).

Surface of Dodecagon $= 11{\cdot}1961524 \times s^2$.

$\odot$ $= 3{\cdot}1415926 \times (\text{radius})^2$.

$\odot^{ce}$ of $\odot$ $= 3{\cdot}1415926 \times (2.\,\text{radius})$.

478. If a, b, c be the sides of any $\triangle$, and $2p$ its perimeter, then its surface is

$$S = \sqrt{\{p\,(p-a).(p-b).(p-c)\}}.$$

If B be the base of a $\triangle$, and P its altitude, then its surface is

$$S = \tfrac{1}{2}P.B.$$

479. If B be the base of a ▱, and P its altitude, then its surface is

$$S = P.B.$$

480. If a quadrilateral be inscriptible in a $\odot$, and a, b, c, d be its sides, and $2p$ is its perimeter, then its surface is

$$S = \sqrt{\{(p-a).(p-b)\,(p-c)\,(p-d)\}}.$$

481. If s be the side or edge of Regular Solids, then (by what has preceded) we get these

Approximations.

Name.	Surface.	Volume.
Tetraedron	$1{\cdot}73205 \times s^2$	$0{\cdot}11785 \times s^3$
Cube	$6 \times s^2$	s^3
Octaedron	$3{\cdot}46410 \times s^2$	$0{\cdot}47140 \times s^3$
Dodecaedron	$20{\cdot}64578 \times s^2$	$7{\cdot}66312 \times s^3$
Icosaedron	$8{\cdot}66025 \times s^2$	$2{\cdot}18169 \times s^3$

482. If R be the radius of a sphere, then we get these approximations relative to it, and the inscribed regular solids, viz.

Name.	Side.	Surface.	Volume.
Sphere	———	$12{\cdot}56637 \times R^2$	$4{\cdot}1879 \times R^3$
Tetraedron	$1{\cdot}62299 \times R$	$4{\cdot}6188 \times R^2$	$0{\cdot}15132 \times R^3$
Cube	$1{\cdot}1547 \times R$	$8 \times R^2$	$1{\cdot}5396 \times R^3$
Octaedron	$1{\cdot}41421 \times R$	$6{\cdot}9282 \times R^2$	$1\frac{1}{3} \times R^3$
Dodecaedron	$0{\cdot}71364 \times R$	$10{\cdot}51462 \times R^2$	$2{\cdot}78516 \times R^3$
Icosaedron	$1{\cdot}05146 \times R$	$9{\cdot}57454 \times R^2$	$2{\cdot}53615 \times R^3$

483. *If four Spheres be given in Position and Magnitude, required to describe another Sphere which shall touch each of them either with its Convex or Concave Surface* (see Carnot, Geom. de Position).

484. *Given, a Sphere whose radius is r, to find* four *other Spheres, = to one another, such that, being placed around the given Sphere, so as to be in contact with it and one another, they shall exactly fill up the space around the given Sphere.*

Answer.—The radius of the required spheres is $(2 + \surd 6) \times r$.

485. *Given, a Sphere whose Radius is r, to find* six *other Spheres, = to one another, such that, being placed about the given Sphere, they may touch it and one another so as to exactly fill up the space about the given Sphere.*

Answer.—The radius of the required spheres is $(1 + \surd 2) \times r$.

486. *Given, a Sphere whose Radius is r, to find* twelve *other = Spheres, such that, being placed about the given Sphere, they may touch it and one another, so as exactly to fill the space around the given Sphere.*

Answer.—The radius of the spheres required is

$$\left\{\frac{3 - \surd 5}{2} + \surd(5 - 2\surd 5)\right\} \times r.$$